Chiral three-nucleon force at N^4LO II: Intermediate-range contributions

H. Krebs,[1, *] A. Gasparyan,[1, 2, †] and E. Epelbaum[1, ‡]

[1] *Institut für Theoretische Physik II, Ruhr-Universität Bochum, D-44780 Bochum, Germany*

[2] *FSBI SSC RF ITEP, Bolshaya Cheremushkinskaya 25, 117218 Moscow, Russia*

(Dated: February 13, 2013)

We derive the subleading contributions to the two-pion-one-pion exchange and ring three-nucleon force topologies emerging at next-to-next-to-next-to-next-to-leading order in chiral effective field theory. The resulting expressions do not involve any unknown parameters. To study convergence of the chiral expansion we work out the most general operator structure of a local isospin-invariant three-nucleon force. Using the resulting operator basis with 22 independent structures, we compare the strength of the corresponding potentials in configuration space for individual topologies at various orders in the chiral expansion. As expected, the subleading contributions from the two-pion-one-pion-exchange and ring diagrams are large which can be understood in terms of intermediate excitation of the $\Delta(1232)$ isobar.

PACS numbers: 13.75.Cs,21.30.-x

I. INTRODUCTION

Three-nucleon forces (3NF) are presently subject to intense research, see Refs. [1–10] for a selection of recent few- and many-body calculations along these lines and Refs. [11, 12] for review articles. On the one hand, rapidly increasing computational resources coupled with sophisticated few- and many-body methods allow nowadays for reliable and accurate nuclear structure calculations for light and even medium-mass nuclei. One can, therefore, relate the properties of the nuclear Hamiltonian to observables in a reliable way and without invoking any uncontrollable approximations. On the other hand, considerable progress has also been reached towards quantitative description of nuclear forces using the framework of chiral effective field theory (EFT), see recent review articles [12–15] and references therein. In particular, nucleon-nucleon (NN) potentials at next-to-next-to-next-to-leading order (N^3LO) in the chiral expansion were developed [16, 17], which allow for an accurate description of NN scattering data up to laboratory energies of the order of $E_{\rm lab} \sim 200$ MeV. For heavier systems, the accuracy of theoretical predictions is currently limited by the 3NFs for which only the dominant contributions at next-to-next-to-leading order (N^2LO) in the chiral expansion of the nuclear Hamiltonian have so far been employed in few- and many-body calculations.

The chiral expansion of the 3NF at one-loop level, i.e. up to next-to-next-to-next-to-next-to-leading order (N^4LO), can be described in terms of six topologies depicted in Fig. 1. The first nonvanishing contributions to the 3NF emerge at N^2LO from tree-level diagrams corresponding to the 2π-exchange, one-pion-exchange-contact and purely contact graphs (a), (d) and (f), respectively [18, 19]. The shorter-range terms emerging from diagrams (d) and (f) depend on one unknown low-energy constant (LEC) each which can be determined from suitable few-nucleon observables, see e.g. [4, 5, 19, 20]. The long-range contribution (a) is, on the other hand, parameter-free since the LECs c_1, c_3 and c_4 accompanying the subleading $\pi\pi NN$ vertices can be extracted from pion-nucleon scattering, see [21–24] for heavy-baryon results, Refs. [25, 26] for some more recent calculations using manifestly covariant formulations of chiral perturbation theory as well as Refs. [27] for an attempt to determine these LECs from proton-proton and neutron-proton partial wave analyses. The resulting 3NF at N^2LO has been intensively explored in three- and four-nucleon scattering calculations, see [11] and references therein. One finds a good description of low-energy nucleon-deuteron scattering observables except for the well-known, long-standing puzzles such as the vector analyzing power in elastic nucleon-deuteron scattering (the so-called A_y-puzzle) and the cross section in the space-star breakup configuration,

*Email: hermann.krebs@rub.de
†Email: ashotg@tp2.rub.de
‡Email: evgeny.epelbaum@rub.de

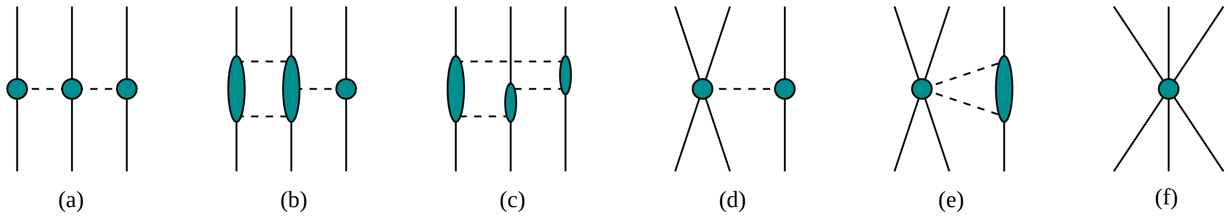

FIG. 1: Various topologies contributing to the 3NF up-to-and-including N^{4}LO: two-pion (2π) exchange (a), two-pion-one-pion (2π-1π) exchange (b), ring (c), one-pion-exchange-contact (d), two-pion-exchange-contact (e) and purely contact (f) diagrams. Solid and dashed lines represent nucleons and pions, respectively. Shaded blobs represent the corresponding amplitudes.

see Ref. [11] for more details. Promising results for low-energy four-nucleon scattering observables based on the chiral 3NF, especially in connection with the A_y-puzzle, are reported in Ref. [3]. While nucleon-deuteron scattering data at higher energies are also well described, the theoretical uncertainty increases rapidly reflecting similar pattern in the two-nucleon sector at this order in the chiral expansion. Promising results based on chiral nuclear forces were also obtained in nuclear structure calculations showing, in particular, sensitivity to the individual terms of the N^{2}LO 3NF, see [11] and references therein. All these findings clearly underline the need to include corrections to the 3NF beyond the leading terms at N^{2}LO.

The first corrections to the 3NF emerge at N^{3}LO from all possible one-loop diagrams of type (a)-(e) involving solely the lowest-order vertices from the effective chiral Lagrangian. The resulting parameter-free expressions for the long-range and intermediate-range contributions of types (a), and (b), (c), respectively, can be found in Ref. [28], see also [29] where the leading one-loop corrections to the longest-range two-pion exchange terms of type (a) are calculated within the infrared-regularized version of chiral perturbation theory. N^{3}LO contributions to shorter-range diagrams of types (d) and (e) as well as the leading relativistic corrections are given in Ref. [30], see also [31] for a related work. Notice that these shorter-range terms are driven by the leading four-nucleon contact operators which also contribute to nucleon-nucleon S-wave scattering. Thus, they do not involve any unknown LECs. Finally, there are no corrections to the purely short-range topology of type (f) at this order. An interesting feature of the N^{3}LO 3NF corrections is their rather rich isospin-spin-momentum structure emerging primarily from the ring topology (c) in Fig. 1. This is in contrast with the quite restricted operator structure of the N^{2}LO 3NF. The impact of these novel 3NF terms on nucleon-deuteron scattering and nuclear structure observables is unknown which makes the complete N^{3}LO calculations even more urgent, especially in connection with the already mentioned unsolved puzzles. Numerical implementation of the new terms in the 3NF at N^{3}LO requires their partial wave decomposition which is a nontrivial task. In Ref. [32], a novel method to perform partial-wave decomposition of any type of the 3NF by carrying out five-dimensional angular integrations numerically was introduced. This approach is quite general in the sense that it can be applied to any type of 3NF but requires substantial computational resources. The partial wave decomposition of the N^{3}LO 3NF using this new technique is in progress, see Ref. [33] for some first (but still incomplete) results.

Meanwhile, one may ask whether the derived expressions for the 3NF at subleading order in the chiral expansion are already converged or, at least, provide a reasonable approximation to the converged result. This applies especially to new operator structures emerging from the genuine loop topologies (b) and (c), whose chiral expansion starts at N^{3}LO rather than N^{2}LO. At this order, the resulting contributions still miss physics associated with intermediate $\Delta(1232)$ excitations. In the standard chiral EFT formulation based on pions and nucleons as the only explicit degrees of freedom, all effects of the Δ (and heavier resonances as well as heavy mesons) are hidden in the (renormalized) values of certain LECs starting from the subleading effective Lagrangian. The major part of the Δ contributions to the nuclear forces is known to be well represented in terms of resonance saturation of the LECs $c_{2,3,4}$ accompanying the subleading $\pi\pi NN$ vertices [21, 34–36] (see, however, the last two references for examples of the Δ-contributions that go beyond the saturation of $c_{2,3,4}$). The values of these LECs are known to be largely driven by the Δ and appear to be large in magnitude. As a consequence, one observes a rather unnatural convergence pattern in the chiral expansion of the two-pion exchange nucleon-nucleon potential $V_{NN}^{2\pi}$ with by far the strongest contribution emerging from the formally subleading triangle diagram proportional to c_3 [37]. The (formally) leading contribution to $V_{NN}^{2\pi}$ does not provide a good approximation to the potential so that one needs to go to higher orders in the chiral expansion and/or include the Δ-isobar as an explicit degree of freedom. One expects similar convergence pattern for the chiral expansion of the 2π-1π exchange and ring 3NF topologies, see also the discussion in Ref. [38]. For the ring topology,

this expectation is in line with the phenomenological study of Ref. [39]. All this suggests that one should not truncate the chiral expansion of the 3NF at N^{3}LO but rather go to (at least) N^{4}LO in the standard Δ-less EFT approach and/or include the Δ-isobar as an explicit degree of freedom. In the latter case, first contributions of the Δ to the 2π-1π exchange and ring 3NF topologies would appear already at N^{3}LO. It should be understood that the strategies outlined above are, to some extent, complementary to each other. This is because N^{4}LO 3NF corrections in the Δ-less theory only take into account (some) effects due to single Δ-excitation but not due to double and triple Δ-excitations which appear first at N^{5}LO and N^{6}LO, respectively. While these effects are included at N^{3}LO in the Δ-full approach, N^{4}LO contributions not related to Δ-excitations are certainly not. We further emphasize that in both cases a number of unknown LECs will appear. It remains to be seen which strategy will turn out to be most efficient in practical terms.

In our recent work [40] we already made a first step in this direction and worked out N^{4}LO corrections to the longest-range 2π-exchange topology in the delta-less approach. Apart from relativistic corrections (which in our power counting scheme appear at N^{3}LO but turn out to vanish at N^{4}LO), the general form of the 2π-exchange 3NF can be parametrized in terms of two scalar functions $\mathcal{A}(q_2)$ and $\mathcal{B}(q_2)$ which depend on the momentum transfer $q_2 \equiv |\vec{q}_2|$ of, say, the second nucleon. In spite of this simple structure, this topology turns out to be most challenging to calculate. The pion-nucleon scattering amplitude enters here at the subleading one-loop order so that the N^{4}LO correction depends not only on the pion decay constant F_π and the pion-nucleon coupling $g_{\pi NN}$ but also on 13 independent (linear combinations of the) LECs from higher-order effective Lagrangians: $c_{1,2,3,4}$ from $\mathcal{L}^{(2)}_{\pi N}$, $\bar{d}_1 + \bar{d}_2$, $\bar{d}_3$, $\bar{d}_5$, $\bar{d}_{14} - \bar{d}_{15}$ from $\mathcal{L}^{(3)}_{\pi N}$ and $\bar{e}_{14,15,16,17,18}$ from $\mathcal{L}^{(4)}_{\pi N}$. The explicit form of the heavy-baryon pion-nucleon effective Lagrangians $\mathcal{L}^{(n)}_{\pi N}$ of chiral dimension n needed in the derivation can be found in [40] while the complete pion-nucleon Lagrangian $\mathcal{L}^{(4)}_{\pi N}$ is constructed in Ref. [41]. In order to determine these LECs we re-analyzed pion-nucleon scattering at subleading one-loop order employing exactly the same power counting scheme as in the derivation of the nuclear forces. We used the available partial wave analyses of the pion-nucleon scattering data to determine all relevant LECs. With all LECs being fixed from pion-nucleon scattering as discussed above, we found a good (reasonable) convergence of the chiral expansion for the functions $\mathcal{A}(q_2)$ ($\mathcal{B}(q_2)$). This is to be expected given that effects of the Δ-isobar are, to a large extent, accounted for already in the leading contribution to $\mathcal{A}(q_2)$ and $\mathcal{B}(q_2)$ at N^{2}LO through resonance saturation of the LECs $c_{3,4}$. As pointed out above, this situation is different for the 2π-1π exchange and ring 3NF topologies, whose leading contributions at N^{3}LO completely miss effects of the Δ-isobar which lets one expect large N^{4}LO corrections.

In the present work we calculate the intermediate-range contributions to the 3NF at N^{4}LO, namely the ones corresponding to diagrams (b) and (c) in Fig. 1, and analyze in detail convergence of the chiral expansion for long-range tail of the 3NF by comparing the coordinate-space potentials associated with individual isospin-spin-position structures. In order to carry out such a comparison in a meaningful way, we worked out the most general structure of a local isospin-invariant 3NF both in momentum and configuration spaces and defined the minimal sets of linearly independent operators. Our paper is organized as follows. In section II we carry out Fourier transformation of the momentum-space expressions for the 2π-exchange 3NF of Ref. [40]. Sections III and IV are devoted to the calculation of the N^{4}LO corrections to the 2π-1π-exchange and ring topologies, respectively. For 2π-1π-exchange contributions we provide results both in momentum and coordinate spaces. For the ring topology we give compact expressions in coordinate space while the rather lengthy expressions in momentum space are delegated to appendix A. The most general operator structure of a local 3NF is worked out in section V where we also define the basis of 22 isospin-spin-momentum operators. We use corresponding coordinate-space version of this basis when discussing numerical results for various potentials in section VI in connection with convergence of the chiral expansion. The findings of our work are briefly summarized in section VII.

II. TWO-PION-EXCHANGE 3NF IN CONFIGURATION SPACE

The 2π-exchange topology (a) generates the longest-range contribution to the 3NF. In the isospin and static limits, i.e. the limit of infinitely heavy nucleons, its general structure in momentum space has the following form (modulo terms of a shorter range corresponding to other topologies):

$$V_{2\pi}(\vec{q}_1,\, \vec{q}_3) = \frac{\vec{\sigma}_1 \cdot \vec{q}_1 \, \vec{\sigma}_3 \cdot \vec{q}_3}{[q_1^2 + M_\pi^2]\,[q_3^2 + M_\pi^2]} \Big(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_3 \, \mathcal{A}(q_2) + \boldsymbol{\tau}_1 \times \boldsymbol{\tau}_3 \cdot \boldsymbol{\tau}_2 \, \vec{q}_1 \times \vec{q}_3 \cdot \vec{\sigma}_2 \, \mathcal{B}(q_2) \Big) \,, \tag{2.1}$$

where M_π stays for the pion mass, $\vec{\sigma}_i$ denote the Pauli spin matrices for the nucleon i and $\vec{q}_i = \vec{p}_i\,' - \vec{p}_i$, with $\vec{p}_i\,'$ and $\vec{p}_i$ being the final and initial momenta of the nucleon i. Here and in what follows, we use the notation: $q_i \equiv |\vec{q}_i|$. Notice

that the momentum transfers are not independent and related to each other via the condition $\vec{q}_1 + \vec{q}_2 + \vec{q}_3 = 0$. The quantities $\mathcal{A}(q_2)$ and $\mathcal{B}(q_2)$ in Eq. (2.1) are scalar functions of the momentum transfer q_2 of the second nucleon whose explicit form is determined by means of the chiral expansion, i.e. the expansion in powers of the soft scale $Q \sim M_\pi$. Unless stated otherwise, the expressions for the 3NF results are always given for a particular choice of the nucleon labels. The complete result can then be found by taking into account all possible permutations of the nucleons

$$V_{\rm 3N}^{\rm full} = V_{\rm 3N} + 5 \text{ permutations}\,. \tag{2.2}$$

The explicit expressions for the functions $\mathcal{A}(q_2)$ and $\mathcal{B}(q_2)$ at first three nonvanishing orders in the chiral expansion, i.e. N^{2}LO $[Q^3]$, N^{3}LO $[Q^4]$ and N^{4}LO $[Q^5]$ [1] are given in Ref. [40]. The functions $\mathcal{A}(q_2)$ and $\mathcal{B}(q_2)$ resulting at different orders in the chiral expansion are plotted versus the values of q_2 in Fig. 5 of that work. While in the case of the 2π-exchange topology it is possible to address the convergence of the chiral expansion in momentum space thanks to the particularly simple parametrization in Eq. (2.1), this is generally not possible for the more complicated cases of the 2π-1π exchange and ring diagrams. This is because there is, in general, no easy way to separate the truly long-range components, which are unambiguously predicted in terms of the chiral expansion, from scheme-dependent short-range contributions. Such a separation is naturally achieved by looking at the corresponding coordinate-space potentials at sufficiently large distances. It is, therefore, advantageous and, in fact, also quite natural to switch to coordinate space in order to study the convergence of the chiral expansion for nuclear forces.

We define the coordinate space representation of a static 3NF by means of the Fourier-transform

$$\tilde{V}_{3N}(\vec{r}_{12},\, \vec{r}_{32}\,) \;=\; \int \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_3}{(2\pi)^3}\, e^{i\vec{q}_1\cdot\vec{r}_{12}}\; e^{i\vec{q}_3\cdot\vec{r}_{32}}\; V_{3N}(\vec{q}_1,\, \vec{q}_3). \tag{2.3}$$

For the two-pion-exchange contribution, we obtain from Eq. (2.1)

$$\tilde{V}_{2\pi}(\vec{r}_{12},\, \vec{r}_{32}\,) = -\vec{\sigma}_1\cdot\vec{\nabla}_{12}\;\vec{\sigma}_3\cdot\vec{\nabla}_{32}\Big(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3\;\tilde{\mathcal{A}}(\vec{r}_{12},\vec{r}_{32}) - \boldsymbol{\tau}_1\times\boldsymbol{\tau}_3\cdot\boldsymbol{\tau}_2\;\vec{\nabla}_{12}\times\vec{\nabla}_{32}\cdot\vec{\sigma}_2\;\tilde{\mathcal{B}}(\vec{r}_{12},\vec{r}_{32})\Big)\,, \tag{2.4}$$

where $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ denotes the distance between the nucleons i and j. The differential operators $\vec{\nabla}_{ij}$ are defined in terms of dimensionless variables $\vec{x}_{ij} = \vec{r}_{ij} M_\pi$; the functions $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are given by

$$\begin{aligned}
\tilde{\mathcal{A}}(\vec{r}_{12},\vec{r}_{32}) \;&=\; \int \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_3}{(2\pi)^3}\, e^{i\vec{q}_1\cdot\vec{r}_{12}}\; e^{i\vec{q}_3\cdot\vec{r}_{32}}\; \frac{1}{q_1^2+M_\pi^2}\frac{1}{q_3^2+M_\pi^2}\;\mathcal{A}(q_2),\\
\tilde{\mathcal{B}}(\vec{r}_{12},\vec{r}_{32}) \;&=\; \int \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_3}{(2\pi)^3}\, e^{i\vec{q}_1\cdot\vec{r}_{12}}\; e^{i\vec{q}_3\cdot\vec{r}_{32}}\; \frac{1}{q_1^2+M_\pi^2}\frac{1}{q_3^2+M_\pi^2}\;\mathcal{B}(q_2).
\end{aligned} \tag{2.5}$$

The N^{2}LO expressions for $\mathcal{A}$ and $\mathcal{B}$ corresponding to $\mathcal{A}^{(3)}(q_2)$ and $\mathcal{B}^{(3)}(q_2)$ from Eq. (3.5) of Ref. [40] are given by

$$\begin{aligned}
\tilde{\mathcal{A}}^{(3)}(\vec{r}_{12},\vec{r}_{32}) \;&=\; \frac{g_A^2 M_\pi^6}{128\pi^2 F_\pi^4}\Big(2c_3 - 4c_1 - c_3(\vec{\nabla}_{12}+\vec{\nabla}_{32})^2\Big)U_1(x_{12})U_1(x_{32}),\\
\tilde{\mathcal{B}}^{(3)}(\vec{r}_{12},\vec{r}_{32}) \;&=\; \frac{g_A^2 M_\pi^6 c_4}{128\pi^2 F_\pi^4}U_1(x_{12})U_1(x_{32}),
\end{aligned} \tag{2.6}$$

where g_A denotes the nucleon axial vector coupling and the Yukawa function U_1 is defined as

$$U_1(x) = \frac{4\pi}{M_\pi}\int \frac{d^3 q}{(2\pi)^3}\frac{e^{i\vec{q}\cdot\vec{x}/M_\pi}}{q^2+M_\pi^2} = \frac{e^{-x}}{x}. \tag{2.7}$$

Here and in what follows, the superscripts of $\mathcal{A}$, $\mathcal{B}$, $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$ as well as other functions parametrizing 3NF matrix elements refer to the chiral dimension, i.e. to the associated power of the soft scale Q.

---

[1] Notice that the overall "chiral dimension" is a matter of convention. In the context of nuclear chiral EFT, one usually uses the convention in which the leading-order (LO) one-pion exchange nucleon-nucleon potential is assigned the chiral dimension Q^0.

The first corrections to $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ emerge from Fourier-transforming the expressions $\mathcal{A}^{(4)}(q_2)$ and $\mathcal{B}^{(4)}(q_2)$ given in Eq. (3.4) of Ref. [40]. We obtain

$$\begin{aligned}
\tilde{\mathcal{A}}^{(4)}(\vec{r}_{12},\vec{r}_{32}) &= \frac{g_A^4 M_\pi^7}{4096\pi^3 F_\pi^6}\Big\{\big[\left(4g_A^2+1\right) - 2\left(g_A^2+1\right)(\vec{\nabla}_{12}+\vec{\nabla}_{32})^2\big] U_1(x_{12})U_1(x_{32}) \\
&+ \frac{1}{4\pi}\Big(2-5(\vec{\nabla}_{12}+\vec{\nabla}_{32})^2+2(\vec{\nabla}_{12}+\vec{\nabla}_{32})^4\Big)\int d^3x\, U_1(|\vec{x}_{12}+\vec{x}|)\, W_1(x)\, U_1(|\vec{x}_{32}+\vec{x}|)\Big\}, \\
\tilde{\mathcal{B}}^{(4)}(\vec{r}_{12},\vec{r}_{32}) &= -\frac{g_A^4 M_\pi^7}{4096\pi^3 F_\pi^6}\Big\{(2g_A^2+1)U_1(x_{12})U_1(x_{32}) \\
&+ \frac{1}{4\pi}\Big(4-(\vec{\nabla}_{12}+\vec{\nabla}_{32})^2\Big)\int d^3x\, U_1(|\vec{x}_{12}+\vec{x}|)\, W_1(x)\, U_1(|\vec{x}_{32}+\vec{x}|)\Big\}.
\end{aligned} \tag{2.8}$$

The profile function W_1 is given in terms of the Fourier-transform of the loop function $A(q)$ appearing in $\mathcal{A}^{(4)}(q_2)$ and $\mathcal{B}^{(4)}(q_2)$:

$$W_1(x) = \frac{4\pi}{M_\pi^2}\int\frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{x}/M_\pi}\, A(q) = \frac{e^{-2x}}{2x^2}\,, \qquad \text{with} \quad A(q) = \frac{1}{2q}\arctan\frac{q}{2M_\pi}\,. \tag{2.9}$$

To give the coordinate space expressions for N^{4}LO contributions we need to Fourier-transform another loop function, namely

$$L(q) = \frac{\sqrt{q^2+4M_\pi^2}}{q}\log\frac{\sqrt{q^2+4M_\pi^2}+q}{2M_\pi}\,, \tag{2.10}$$

which enters the expressions for $\mathcal{A}^{(5)}(q_2)$ and $\mathcal{B}^{(5)}(q_2)$ in Eq. (3.14) of Ref. [40]. This can be most easily achieved by using the spectral representation of L given by

$$L(q) = 1 + \int_{2M_\pi}^\infty d\mu \frac{q^2}{\mu^2+q^2}\frac{1}{\mu^2}\sqrt{\mu^2-4M_\pi^2}. \tag{2.11}$$

The Fourier-transform of the square-integrable part of L is given by

$$V_1(x) = \frac{4\pi}{M_\pi}\int\frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{x}/M_\pi}\int_{2M_\pi}^\infty d\mu\frac{1}{\mu^2+q^2}\frac{1}{\mu^2}\sqrt{\mu^2-4M_\pi^2} = \frac{1}{x}\int_2^\infty d\mu\frac{e^{-x\,\mu}}{\mu^2}\sqrt{\mu^2-4}. \tag{2.12}$$

With these preparations, we obtain the following result for the Fourier-transform of $\mathcal{A}^{(5)}(q_2)$ and $\mathcal{B}^{(5)}(q_2)$:

$$\begin{aligned}
\tilde{\mathcal{A}}^{(5)}(\vec{r}_{12},\vec{r}_{32}) &= \frac{g_A M_\pi^8}{73728\pi^4 F_\pi^6}\Big[-(\vec{\nabla}_{12}+\vec{\nabla}_{32})^2\big(F_\pi^2\left(2304\pi^2 g_A(4\bar{e}_{14}+2\bar{e}_{19}-\bar{e}_{22}-\bar{e}_{36})-2304\pi^2\bar{d}_{18}c_3\right) \\
&+ g_A(144c_1-53c_2-90c_3)\big) + F_\pi^2\left(4608\pi^2\bar{d}_{18}(2c_1-c_3)+4608\pi^2 g_A(2\bar{e}_{14}+2\bar{e}_{19}-\bar{e}_{36}-4\bar{e}_{38})\right) \\
&+ g_A\left(72\left(64\pi^2\bar{l}_3+1\right)c_1-24c_2-36c_3\right) \\
&+ (\vec{\nabla}_{12}+\vec{\nabla}_{32})^4\left(2304\pi^2\bar{e}_{14}F_\pi^2 g_A-2g_A(5c_2+18c_3)\right)\Big]U_1(x_{12})U_1(x_{32}) \\
&- \frac{g_A^2 M_\pi^8}{12288\pi^4 F_\pi^6}\Big(1-2(\vec{\nabla}_{12}+\vec{\nabla}_{32})^2\Big)\Big(4(6c_1-c_2-3c_3)-(\vec{\nabla}_{12}+\vec{\nabla}_{32})^2(-c_2-6c_3)\Big)U_1(x_{12})U_1(x_{32}) \\
&+ \frac{g_A^2 M_\pi^8}{49152\pi^5 F_\pi^6}\Big(1-2(\vec{\nabla}_{12}+\vec{\nabla}_{32})^2\Big)\Big(4(6c_1-c_2-3c_3) \\
&- (\vec{\nabla}_{12}+\vec{\nabla}_{32})^2(-c_2-6c_3)\Big)(\vec{\nabla}_{12}+\vec{\nabla}_{32})^2\int d^3x\, U_1(|\vec{x}_{12}+\vec{x}|)\, V_1(x)\, U_1(|\vec{x}_{32}+\vec{x}|), \\
\tilde{\mathcal{B}}^{(5)}(\vec{r}_{12},\vec{r}_{32}) &= -\frac{g_A M_\pi^8}{36864\pi^4 F_\pi^6}\Big[F_\pi^2\left(1152\pi^2\bar{d}_{18}c_4-1152\pi^2 g_A(2\bar{e}_{17}+2\bar{e}_{21}-\bar{e}_{37})\right)+108g_A^3c_4+24g_Ac_4 \\
&- (\vec{\nabla}_{12}+\vec{\nabla}_{32})^2\left(5g_Ac_4-1152\pi^2\bar{e}_{17}F_\pi^2 g_A\right)\Big]U_1(x_{12})U_1(x_{32}) \\
&+ \frac{g_A^2 c_4 M_\pi^8}{6144\pi^4 F_\pi^6}\Big(4-(\vec{\nabla}_{12}+\vec{\nabla}_{32})^2\Big)U_1(x_{12})U_1(x_{32}) \\
&- \frac{g_A^2 c_4 M_\pi^8}{24576\pi^5 F_\pi^6}\Big(4-(\vec{\nabla}_{12}+\vec{\nabla}_{32})^2\Big)(\vec{\nabla}_{12}+\vec{\nabla}_{32})^2\int d^3x\, U_1(|\vec{x}_{12}+\vec{x}|)\, V_1(x)\, U_1(|\vec{x}_{32}+\vec{x}|).
\end{aligned} \tag{2.13}$$

It remains to emphasize that while the momentum space representation of the functions $\mathcal{A}$ and $\mathcal{B}$ depends on just one variable q_2, the coordinate-space functions $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ depend on three scalar arguments. We will discuss the convergence of the chiral expansion for the coordinate space potentials in section VI.

III. TWO-PION-ONE-PION EXCHANGE 3NF AT N^{4}LO

We now turn to the 2π-1π exchange topology. In contrast to the longest-range 2π exchange topology discussed in the previous section, its chiral expansion starts at N^{3}LO. At this order, one has to evaluate all one-loop diagrams made out of the lowest-order pion-nucleon vertices. This was achieved in Ref. [28], see Eqs. (2.16)-(2.23) of that work. As pointed out in Ref. [40], the decomposition of momentum-space 3NF expressions according to the type of the topology is not unique as e.g. some parts of the 2π exchange contributions can be reshuffled into 2π-1π exchange and shorter-range terms by canceling pion propagators with the corresponding expressions in the numerator. In Ref. [40] we introduced a "minimal" parametrization of the 2π exchange 3NF which corresponds to Eq. (2.1) and which is adopted here and in what follows.

The structure of two-pion-one-pion exchange contributions up to N^{4}LO in the chiral expansion has the form

$$
\begin{aligned}
V_{2\pi\text{-}1\pi} &= \frac{\vec{\sigma}_3\cdot\vec{q}_3}{q_3^2+M_\pi^2}\Big[\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3\ [\vec{\sigma}_2\cdot\vec{q}_1\ \vec{q}_1\cdot\vec{q}_3\ F_1(q_1)+\vec{\sigma}_2\cdot\vec{q}_1\ F_2(q_1)+\vec{\sigma}_2\cdot\vec{q}_3\ F_3(q_1)]+\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\ [\vec{\sigma}_1\cdot\vec{q}_1\ \vec{q}_1\cdot\vec{q}_3\ F_4(q_1)\\
&\quad+\vec{\sigma}_1\cdot\vec{q}_3\ F_5(q_1)+\vec{\sigma}_2\cdot\vec{q}_1\ \vec{q}_1\cdot\vec{q}_3\ F_6(q_1)+\vec{\sigma}_2\cdot\vec{q}_1\ F_7(q_1)+\vec{\sigma}_2\cdot\vec{q}_3\ \vec{q}_1\cdot\vec{q}_3\ F_8(q_1)+\vec{\sigma}_2\cdot\vec{q}_3\ F_9(q_1)]\\
&\quad+\boldsymbol{\tau}_1\times\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\,[\vec{\sigma}_1\times\vec{\sigma}_2\cdot\vec{q}_1\ (\vec{q}_1\cdot\vec{q}_3\ F_{10}(q_1)+F_{11}(q_1))+\vec{q}_1\times\vec{q}_3\cdot\vec{\sigma}_1\ \vec{q}_1\cdot\vec{\sigma}_2\ F_{12}(q_1)]\Big]\,,
\end{aligned}
\tag{3.14}
$$

where $F_{1\ldots12}(q_1)$ are scalar functions to be calculated. Notice that we use a slightly different notation compared to our early paper [28], which is now also valid at N^{4}LO. First non-vanishing contributions to the structure functions F_i are generated at N^{3}LO by diagrams shown in Fig. 3 of that work. Adjusting the expressions obtained in Ref. [28] to our new notation and taking into account terms induced by reshuffling the 2π exchange contributions as explained above we obtain the following results for the functions $F_i(q_1)$:

$$
\begin{aligned}
F_1^{(4)}(q_1) &= \frac{g_A^4}{256\pi F_\pi^6 q_1^2}\Big[A(q_1)\left(\left(8g_A^2-4\right)M_\pi^2+\left(g_A^2+1\right)q_1^2\right)-\frac{M_\pi}{4M_\pi^2+q_1^2}\left(\left(8g_A^2-4\right)M_\pi^2+\left(3g_A^2-1\right)q_1^2\right)\Big],\\
F_2^{(4)}(q_1) &= \frac{g_A^4}{128\pi F_\pi^6}A(q_1)\left(2M_\pi^2+q_1^2\right),\\
F_3^{(4)}(q_1) &= -\frac{g_A^4}{256\pi F_\pi^6}A(q_1)\left(\left(8g_A^2-4\right)M_\pi^2+\left(3\ g_A^2-1\right)q_1^2\right),\\
F_4^{(4)}(q_1) &= -\frac{F_5^{(4)}(q_1)}{q_1^2}=-\frac{g_A^6}{128\pi F_\pi^6}A(q_1),\\
F_6^{(4)}(q_1) &= F_8^{(4)}(q_1)=F_9^{(4)}(q_1)=F_{10}^{(4)}(q_1)=F_{12}^{(4)}(q_1)=0,\\
F_7^{(4)}(q_1) &= \frac{g_A^4}{128\pi F_\pi^6}A(q_1)\left(2M_\pi^2+q_1^2\right),\\
F_{11}^{(4)}(q_1) &= -\frac{g_A^4}{512\pi F_\pi^6}A(q_1)\left(4M_\pi^2+q_1^2\right).
\end{aligned}
\tag{3.15}
$$

Notice that we give here only non-polynomial parts as the polynomial ones simply lead to shifts of the low-energy constants D and E from N^{2}LO three-body force.

First corrections to these results emerge at N^{4}LO from diagrams shown in Fig. 2, which involve a single insertion of c_i-vertices from the subleading pion-nucleon Lagrangian. Evaluating the irreducible contributions of these diagrams following the lines of Refs. [28, 30] and keeping only terms non-polynomial in q_1 we obtain the following expressions:

$$
\begin{aligned}
F_1^{(5)}(q_1) &= -\frac{g_A^2c_4}{96\pi^2F_\pi^6q_1^2\left(4M_\pi^2+q_1^2\right)}L(q_1)\left(8\left(4g_A^2-1\right)M_\pi^4+2\left(5g_A^2+1\right)M_\pi^2q_1^2-\left(g_A^2-1\right)q_1^4\right)\\
&\quad-\ \frac{\left(1-4g_A^2\right)g_A^2c_4M_\pi^2}{48\pi^2F_\pi^6q_1^2},
\end{aligned}
$$

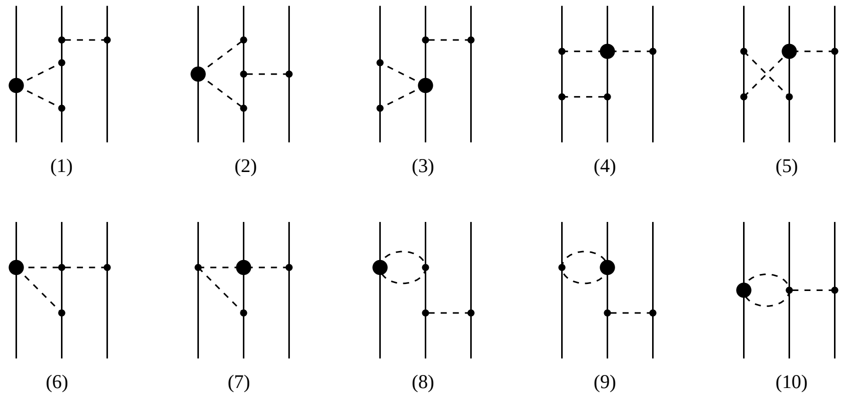

FIG. 2: Two-pion-one-pion exchange 3N diagrams at N^{4}LO. Solid dots and filled circles denote the leading and subleading pion-nucleon vertices, respectively. Graphs resulting from the interchange of the nucleon lines and/or applying time reversal operation are not shown. For remaining notation see Fig. 1.

$$
\begin{aligned}
F_2^{(5)}(q_1) &= F_8^{(5)}(q_1) = F_{11}^{(5)}(q_1) = 0, \\
F_3^{(5)}(q_1) &= -\frac{g_A^2 c_4}{48\pi^2 F_\pi^6 \left(4M_\pi^2 + q_1^2\right)} L(q_1) \left(4\left(4g_A^2 - 1\right) M_\pi^4 + \left(17g_A^2 - 5\right) M_\pi^2 q_1^2 + \left(4g_A^2 - 1\right) q_1^4\right), \\
F_4^{(5)}(q_1) &= -\frac{F_5^{(5)}(q_1)}{q_1^2} = -\frac{g_A^4 c_4}{16\pi^2 F_\pi^6} L(q_1), \\
F_6^{(5)}(q_1) &= \frac{g_A^4 M_\pi^2 (6c_1 + c_2 - 3c_3)}{96\pi^2 F_\pi^6 q_1^2} \\
&+ \frac{g_A^4 L(q_1)}{192\pi^2 F_\pi^6 q_1^2 \left(4M_\pi^2 + q_1^2\right)} \left(-48c_1 M_\pi^4 + c_2\left(-8M_\pi^4 + 2M_\pi^2 q_1^2 + q_1^4\right) + 12c_3 M_\pi^2 \left(2M_\pi^2 + q_1^2\right)\right), \\
F_7^{(5)}(q_1) &= -\frac{g_A^2}{192\pi^2 F_\pi^6} L(q_1) \left(24c_1 M_\pi^2 - c_2\left(4M_\pi^2 + q_1^2\right) - 6c_3\left(2M_\pi^2 + q_1^2\right)\right), \\
F_9^{(5)}(q_1) &= -\frac{g_A^4 L(q_1)}{128\pi^2 F_\pi^6 \left(4M_\pi^2 + q_1^2\right)} \Big[-32c_1 M_\pi^2 \left(3M_\pi^2 + q_1^2\right) + c_2\left(16M_\pi^4 + 16M_\pi^2 q_1^2 + 3q_1^4\right) \\
&+ c_3\left(80M_\pi^4 + 68M_\pi^2 q_1^2 + 13q_1^4\right)\Big], \\
F_{10}^{(5)}(q_1) &= F_{12}^{(5)}(q_1) = \frac{g_A^4 c_4 L(q_1)}{64\pi^2 F_\pi^6}.
\end{aligned} \tag{3.16}
$$

It is straightforward to transform these results into coordinate space. The general structure corresponding to the momentum-space expression in Eq. (3.14) has the form:

$$
\begin{aligned}
\tilde{V}_{2\pi\text{-}1\pi}(\vec{r}_{12}, \vec{r}_{32}) &= \vec{\sigma}_3 \cdot \vec{\nabla}_{32} \Big[\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_3 \,\Big(\vec{\sigma}_2 \cdot \vec{\nabla}_{12}\, \vec{\nabla}_{12} \cdot \vec{\nabla}_{32}\, \tilde{F}_1(x_{12}) - \vec{\sigma}_2 \cdot \vec{\nabla}_{12}\, \tilde{F}_2(x_{12}) - \vec{\sigma}_2 \cdot \vec{\nabla}_{32}\, \tilde{F}_3(x_{12})\Big) \\
&+ \boldsymbol{\tau}_2 \cdot \boldsymbol{\tau}_3 \,\Big(\vec{\sigma}_1 \cdot \vec{\nabla}_{12}\, \vec{\nabla}_{12} \cdot \vec{\nabla}_{32}\, \tilde{F}_4(x_{12}) - \vec{\sigma}_1 \cdot \vec{\nabla}_{32}\, \tilde{F}_5(x_{12}) + \vec{\sigma}_2 \cdot \vec{\nabla}_{12}\, \vec{\nabla}_{12} \cdot \vec{\nabla}_{32}\, \tilde{F}_6(x_{12}) \\
&- \vec{\sigma}_2 \cdot \vec{\nabla}_{12}\, \tilde{F}_7(x_{12}) + \vec{\sigma}_2 \cdot \vec{\nabla}_{32}\, \vec{\nabla}_{12} \cdot \vec{\nabla}_{32}\, \tilde{F}_8(x_{12}) - \vec{\sigma}_2 \cdot \vec{\nabla}_{32}\, \tilde{F}_9(x_{12})\Big) \\
&+ \boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2 \cdot \boldsymbol{\tau}_3 \Big(\vec{\sigma}_1 \times \vec{\sigma}_2 \cdot \vec{\nabla}_{12}\, (\vec{\nabla}_{12} \cdot \vec{\nabla}_{32}\, \tilde{F}_{10}(x_{12}) - \tilde{F}_{11}(x_{12})) \\
&+ \vec{\nabla}_{12} \times \vec{\nabla}_{32} \cdot \vec{\sigma}_1\, \vec{\nabla}_{12} \cdot \vec{\sigma}_2 \tilde{F}_{12}(x_{12})\Big)\Big]\, U_1(x_{32})\,.
\end{aligned} \tag{3.17}
$$

In order to calculate the functions F_i it is convenient to employ the spectral representations of the function $L(q)$, see

Eq. (2.11), and $A(q)$ given by

$$A(q) = \frac{1}{2} \int_{2M_\pi}^{\infty} d\mu \frac{1}{\mu^2 + q^2}. \tag{3.18}$$

Following Ref. [28], we define the profile function $W_3(x)$ via

$$W_3(x) = \frac{4\pi}{M_\pi^2} \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{x}/M_\pi} \left[\frac{M_\pi}{q^2} - \frac{4M_\pi^2}{q^2} A(q)\right] = 2Ei(-2x) + \frac{e^{-2x}}{x}\,, \tag{3.19}$$

where

$$Ei(x) \equiv -\int_{-x}^{\infty} \frac{e^{-t}\, dt}{t}\,. \tag{3.20}$$

Fourier transform of terms in Eq. (3.16) involving the function $L(q)$ can be expressed using the profile functions $V_1(x)$ from Eq. (2.12) and $V_2(x)$ which is defined according to

$$V_2(x) = \frac{1}{x} \int_2^{\infty} d\mu \frac{e^{-x\,\mu}}{\mu^2 \sqrt{\mu^2 - 4}}\,. \tag{3.21}$$

With these definitions, the N^{3}LO contributions to the $\tilde{F}_i$-functions are given by

$$\begin{aligned}
\tilde{F}_1^{(4)}(x_{12}) &= -\frac{g_A^4 M_\pi^7}{4096\pi^3 F_\pi^6} \left(2g_A^2 U_1(2x_{12}) - \left(g_A^2+1\right) W_1(x_{12}) + \left(2g_A^2 - 1\right) W_3(x_{12})\right),\\
\tilde{F}_2^{(4)}(x_{12}) &= -\frac{g_A^4 M_\pi^7}{2048\pi^3 F_\pi^6} \left(\nabla_{12}^2 - 2\right) W_1(x_{12}),\\
\tilde{F}_3^{(4)}(x_{12}) &= \frac{g_A^4 M_\pi^7}{4096\pi^3 F_\pi^6} \left(-\nabla_{12}^2 + \left(3\nabla_{12}^2 - 8\right) g_A^2 + 4\right) W_1(x_{12}),\\
\tilde{F}_4^{(4)}(x_{12}) &= -\frac{g_A^6 M_\pi^7}{2048\pi^3 F_\pi^6} W_1(x_{12}),\\
\tilde{F}_5^{(4)}(x_{12}) &= -\frac{g_A^6 M_\pi^7}{2048\ \pi^3 F_\pi^6} \nabla_{12}^2 W_1(x_{12}),\\
\tilde{F}_6^{(4)}(x_{12}) &= \tilde{F}_8^{(4)}(x_{12}) = \tilde{F}_9^{(4)}(x_{12}) = \tilde{F}_{10}^{(4)}(x_{12}) = \tilde{F}_{12}^{(4)}(x_{12}) = 0,\\
\tilde{F}_7^{(4)}(x_{12}) &= -\frac{g_A^4 M_\pi^7}{2048\pi^3 F_\pi^6} \left(\nabla_{12}^2 - 2\right)\ W_1(x_{12}),\\
\tilde{F}_{11}^{(4)}(x_{12}) &= \frac{g_A^4 M_\pi^7}{8192\pi^3 F_\pi^6} \left(\nabla_{12}^2 - 4\right)\ W_1(x_{12}),
\end{aligned} \tag{3.22}$$

while for the N^{4}LO contributions we obtain the following results:

$$\begin{aligned}
\tilde{F}_1^{(5)}(x_{12}) &= \frac{g_A^2 c_4 M_\pi^8}{3072\pi^4 F_\pi^6} \left[\left(\nabla_{12}^2 - 4\right)\left(-\nabla_{12}^2 + \left(\nabla_{12}^2 + 10\right) g_A^2 + 2\right) U_1(2x_{12})\right.\\
&- \left. 2\left(-\nabla_{12}^4 + 2\nabla_{12}^2 + \left(\nabla_{12}^4 + 10\nabla_{12}^2 - 32\right) g_A^2 + 8\right) V_2(x_{12})\right],\\
\tilde{F}_2^{(5)}(x_{12}) &= \tilde{F}_8^{(5)}(x_{12}) = \tilde{F}_{11}^{(5)}(x_{12}) = 0,\\
\tilde{F}_3^{(5)}(x_{12}) &= \frac{g_A^2 c_4 M_\pi^8}{1536\pi^4 F_\pi^6} \left(-\nabla_{12}^4 + 5\nabla_{12}^2 + \left(4\nabla_{12}^4 - 17\nabla_{12}^2 + 16\right) g_A^2 - 4\right)\\
&\times \left(\left(\nabla_{12}^2 - 4\right) U_1(2x_{12}) - 2\nabla_{12}^2 V_2(x_{12})\right),\\
\tilde{F}_4^{(5)}(x_{12}) &= \frac{g_A^4 c_4 M_\pi^8}{256\pi^4 F_\pi^6} \nabla_{12}^2 V_1(x_{12}),\\
\tilde{F}_5^{(5)}(x_{12}) &= \frac{g_A^4 c_4 M_\pi^8}{256\pi^4 F_\pi^6} \nabla_{12}^4 V_1(x_{12}),\\
\tilde{F}_6^{(5)}(x_{12}) &= \frac{g_A^4 M_\pi^8}{6144\pi^4 F_\pi^6} \left[2\left(48c_1 + \left(-\nabla_{12}^4 + 2\nabla_{12}^2 + 8\right) c_2 + 12\left(\nabla_{12}^2 - 2\right) c_3\right) V_2(x_{12})\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \;\; \left(\nabla_{12}^2-4\right)\left(\left(\nabla_{12}^2-2\right)c_2-12c_3\right)U_1(2x_{12})\big] \,, \\
\tilde{F}_7^{(5)}(x_{12}) \;&=\; \frac{g_A^2 M_\pi^8}{3072\pi^4 F_\pi^6}\nabla_{12}^2\left(24c_1+\left(\nabla_{12}^2-4\right)c_2+6\left(\nabla_{12}^2-2\right)c_3\right)V_1(x_{12}), \\
\tilde{F}_9^{(5)}(x_{12}) \;&=\; \frac{g_A^4 M_\pi^8}{4096\pi^4 F_\pi^6}\left(32\left(\nabla_{12}^2-3\right)c_1+\left(3\nabla_{12}^4-16\nabla_{12}^2+16\right)c_2+\left(13\nabla_{12}^4-68\nabla_{12}^2+80\right)c_3\right) \\
&\times \;\; \left(\left(\nabla_{12}^2-4\right)U_1(2x_{12})-2\nabla_{12}^2 V_2(x_{12})\right), \\
\tilde{F}_{10}^{(5)}(x_{12}) \;&=\; \tilde{F}_{12}^{(5)}(x_{12}) \;=\; -\frac{g_A^4 c_4 M_\pi^8}{1024\pi^4 F_\pi^6}\nabla_{12}^2 V_1(x_{12}).
\end{aligned}
\tag{3.23}
$$

IV. RING DIAGRAMS AT N^{4}LO

Finally, we consider the ring topology. The leading contributions emerge at N^{3}LO from diagrams shown in Fig. 4 of Ref. [28]. As explained in that paper, only diagrams proportional to g_A^6 and g_A^4 generate nonvanishing 3NFs:

$$
V_{\rm ring}^{(4)} \;=\; V_{\rm ring}^{(4),g_A^6}+V_{\rm ring}^{(4),g_A^4}. \tag{4.24}
$$

Evaluating the corresponding loop integrals in momentum space we obtained complicated expressions involving three-point function which are given explicitly in appendix of Ref. [28].[2] The results in coordinate space are much more compact and have the form:

$$
\begin{aligned}
V_{\rm ring}^{(4),g_A^6}(\vec r_{12},\,\vec r_{32}\,) \;&=\; \left(\frac{g_A}{2F_\pi}\right)^6\int\frac{d^3l_1}{(2\pi)^3}\,\frac{d^3l_2}{(2\pi)^3}\,\frac{d^3l_3}{(2\pi)^3}\,e^{i\vec l_1\cdot\vec r_{23}}\,e^{i\vec l_2\cdot\vec r_{31}}\,e^{i\vec l_3\cdot\vec r_{12}}\,\frac{v}{[l_1^2+M_\pi^2]\,[l_2^2+M_\pi^2]^2\,[l_3^2+M_\pi^2]} \\
&=\; -\frac{g_A^6\,M_\pi^7}{4096\,\pi^3\,F_\pi^6}\Big[-4\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\,\vec\nabla_{23}\times\vec\nabla_{12}\cdot\vec\sigma_2\,\vec\nabla_{23}\times\vec\nabla_{31}\cdot\vec\sigma_3\,\vec\nabla_{31}\cdot\vec\nabla_{12} \\
&\quad -\;2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3\,\vec\nabla_{23}\cdot\vec\nabla_{31}\,\vec\nabla_{23}\cdot\vec\nabla_{12}\,\vec\nabla_{31}\cdot\vec\nabla_{12}\;+\;\boldsymbol{\tau}_1\times\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\,\vec\nabla_{23}\times\vec\nabla_{12}\cdot\vec\sigma_2\,\vec\nabla_{23}\cdot\vec\nabla_{31}\,\vec\nabla_{31}\cdot\vec\nabla_{12} \\
&\quad +\;3\vec\nabla_{31}\times\vec\nabla_{12}\cdot\vec\sigma_1\,\vec\nabla_{23}\times\vec\nabla_{31}\cdot\vec\sigma_3\,\vec\nabla_{23}\cdot\vec\nabla_{12}\Big]\,U_1(x_{23})\,U_2(x_{31})\,U_1(x_{12})\,, \\
V_{\rm ring}^{(4),g_A^4}(\vec r_{12},\,\vec r_{32}\,) \;&=\; \frac{g_A^4\,M_\pi^7}{2048\,\pi^3\,F_\pi^6}\Big[2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\left(\vec\nabla_{23}\cdot\vec\nabla_{31}\,\vec\nabla_{31}\cdot\vec\nabla_{12}-\vec\nabla_{31}\times\vec\nabla_{12}\cdot\vec\sigma_1\,\vec\nabla_{23}\times\vec\nabla_{31}\cdot\vec\sigma_3\right) \\
&\quad +\boldsymbol{\tau}_1\times\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\,\vec\nabla_{31}\times\vec\nabla_{12}\cdot\vec\sigma_1\,\vec\nabla_{23}\cdot\vec\nabla_{31}\Big]U_1(x_{23})\,U_1(x_{31})\,U_1(x_{12}),
\end{aligned}
\tag{4.26}
$$

where the derivatives should be evaluated as if the variables $\vec x_{12}$, $\vec x_{23}$ and $\vec x_{31}$ were independent[3] and the numerator v in the first line is given by

$$
\begin{aligned}
v \;=\; &-8\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\,\vec l_1\times\vec l_3\cdot\vec\sigma_2\,\vec l_1\times\vec l_2\cdot\vec\sigma_3\,\vec l_2\cdot\vec l_3\;-\;4\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3\,\vec l_1\cdot\vec l_2\,\vec l_1\cdot\vec l_3\,\vec l_2\cdot\vec l_3\;+\;2\boldsymbol{\tau}_1\times\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\,\vec l_1\times\vec l_3\cdot\vec\sigma_2\,\vec l_1\cdot\vec l_2\,\vec l_2\cdot\vec l_3 \\
&+6\vec l_2\times\vec l_3\cdot\vec\sigma_1\,\vec l_1\times\vec l_2\cdot\vec\sigma_3\,\vec l_1\cdot\vec l_3\,.
\end{aligned}
\tag{4.27}
$$

At N^{4}LO, one only needs to evaluate the contributions of the four diagrams shown in Fig. 3,

$$
V_{\rm ring}^{(5)} \;=\; V_{\rm ring}^{(5),g_A^4}+V_{\rm ring}^{(5),g_A^2}+V_{\rm ring}^{(5),g_A^0}\,. \tag{4.28}
$$

---

[2] We emphasize that several symmetry factors are missing in Eq. (A1) of that work. The corrected equation has the form:

$$
\begin{aligned}
V_{\rm ring} \;&=\; \vec\sigma_1\cdot\vec\sigma_2\,\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\,R_1+\vec\sigma_1\cdot\vec q_1\vec\sigma_2\cdot\vec q_1\,\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\,R_2+\vec\sigma_1\cdot\vec q_1\vec\sigma_2\cdot\vec q_3\,\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\,R_3+\vec\sigma_1\cdot\vec q_3\vec\sigma_2\cdot\vec q_1\,\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\,R_4 \\
&+\;\vec\sigma_1\cdot\vec q_3\vec\sigma_2\cdot\vec q_3\,\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\,R_5+\frac{1}{2}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3\,R_6+\vec\sigma_1\cdot\vec q_1\vec\sigma_3\cdot\vec q_1\,R_7+\frac{1}{2}\vec\sigma_1\cdot\vec q_1\vec\sigma_3\cdot\vec q_3\,R_8+\frac{1}{2}\vec\sigma_1\cdot\vec q_3\vec\sigma_3\cdot\vec q_1\,R_9 \\
&+\;\frac{1}{2}\vec\sigma_1\cdot\vec\sigma_3\,R_{10}+\frac{1}{2}\vec q_1\cdot\vec q_3\times\vec\sigma_2\,\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\times\boldsymbol{\tau}_3\,R_{11}.
\end{aligned}
\tag{4.25}
$$

[3] Clearly, the relative distances $\vec r_{12}$, $\vec r_{23}$ and $\vec r_{31}$ are related via $\vec r_{12}+\vec r_{23}+\vec r_{31}=0$.

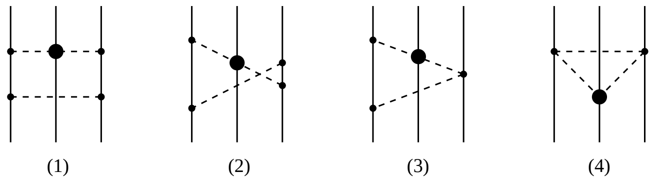

FIG. 3: Ring diagrams at N^{4}LO. For notation see Figs. 1, 2.

We were again able to obtain fairly compact expressions in coordinate space, which, however, involve now a single scalar integral over the mass of the exchanged particles:

$$
\begin{aligned}
V_{\rm ring}^{(5),g_A^4} &= -\frac{g_A^4 M_\pi^8}{1024\pi^4 F_\pi^6}\int_{-\infty}^{\infty} ds \Big[2\vec{\nabla}_{12}\cdot\vec{\nabla}_{23}\Big(\vec{\nabla}_{23}\cdot\vec{\nabla}_{31}\left(12c_1\vec{\sigma}_2\cdot\vec{\sigma}_3 - 4c_2 s^2\vec{\sigma}_2\cdot\vec{\sigma}_3\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3 + 6c_2 s^2\right.\\
&+ \vec{\nabla}_{12}\cdot\vec{\nabla}_{31}(\vec{\sigma}_1\cdot\vec{\sigma}_3(-3c_3 + c_4\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + c_4\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3) + 2(\vec{\sigma}_2\cdot\vec{\sigma}_3(-3c_3 + c_4\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + c_4\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3)\\
&+ c_3(2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3))) + 2\vec{\nabla}_{12}\cdot\vec{\sigma}_1\vec{\nabla}_{31}\cdot\vec{\sigma}_2(3c_3 - c_4\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3 - c_4\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3) - 4c_3 s^2\vec{\sigma}_2\cdot\vec{\sigma}_3\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\\
&+ \left.6c_3 s^2 + c_4\vec{\nabla}_{12}\cdot\vec{\nabla}_{31}\times\vec{\sigma}_1\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\times\boldsymbol{\tau}_3\right) - 2\Big(2\vec{\nabla}_{12}\cdot\vec{\sigma}_1\vec{\nabla}_{31}\cdot\vec{\sigma}_2\left(3c_1 - s^2(c_2+c_3)\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\right)\\
&+ \left(\vec{\nabla}_{31}\cdot\vec{\sigma}_1\vec{\nabla}_{31}\cdot\vec{\sigma}_3 - \left(s^2+1\right)\vec{\sigma}_1\cdot\vec{\sigma}_3\right)\left(s^2(c_2+c_3)\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3 - 3c_1\right) + \vec{\nabla}_{12}\cdot\vec{\nabla}_{31}(4c_1\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\\
&+ \vec{\nabla}_{23}\cdot\vec{\sigma}_1\vec{\nabla}_{31}\cdot\vec{\sigma}_3(-3c_3 + c_4\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + c_4\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3))\Big) - \left(s^2+1\right)\vec{\nabla}_{12}\cdot\vec{\sigma}_3\vec{\nabla}_{23}\cdot\vec{\sigma}_1(3c_3 - c_4\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\\
&- c_4\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3)) + \vec{\nabla}_{12}\cdot\vec{\nabla}_{31}\Big(8\Big(\vec{\nabla}_{12}\cdot\vec{\sigma}_2\vec{\nabla}_{23}\cdot\vec{\sigma}_1\left(s^2(c_2+c_3)\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 - 3c_1\right)\\
&+ \vec{\nabla}_{23}\cdot\vec{\sigma}_1\vec{\nabla}_{31}\cdot\vec{\sigma}_3\left(s^2(c_2+c_3)\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3 - 3c_1\right) + \left(s^2+1\right)\vec{\sigma}_2\cdot\vec{\sigma}_3\left(s^2(c_2+c_3)\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3 - 3c_1\right)\Big)\\
&- \vec{\nabla}_{23}\cdot\vec{\nabla}_{31}\Big(4\vec{\sigma}_1\cdot\vec{\sigma}_3\left(s^2(c_2+c_3)\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3 - 3c_1\right) + 8c_1\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3 - 6c_2 s^2 + 4\vec{\nabla}_{12}\cdot\vec{\sigma}_2\vec{\nabla}_{23}\cdot\vec{\sigma}_1\\
&\times (-3c_3 + c_4\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3 + c_4\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3) - 6c_3 s^2 + c_4\vec{\nabla}_{12}\cdot\vec{\nabla}_{23}\times\vec{\sigma}_2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\times\boldsymbol{\tau}_3\Big)\Big)\\
&+ 4\Big(\left(s^2+1\right)\vec{\nabla}_{23}\cdot\vec{\sigma}_1\Big(\vec{\nabla}_{31}\cdot\vec{\sigma}_2\left(6c_1 - 2s^2(c_2+c_3)\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \vec{\nabla}_{23}\cdot\vec{\nabla}_{31}(-3c_3 + c_4\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3 + c_4\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3)\right)\\
&+ \vec{\nabla}_{12}\cdot\vec{\sigma}_3\left(3c_1 - s^2(c_2+c_3)\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3\right)\Big) + \vec{\nabla}_{12}\cdot\vec{\sigma}_1\vec{\nabla}_{12}\cdot\vec{\sigma}_2\vec{\nabla}_{23}\cdot\vec{\nabla}_{31}\left(6c_1 - 2s^2(c_2+c_3)\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\right.\\
&+ \left.\vec{\nabla}_{23}\cdot\vec{\nabla}_{31}(-3c_3 + c_4\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3 + c_4\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3)\right)\Big)\Big) - 2(\vec{\nabla}_{12}\cdot\vec{\nabla}_{23})^2\left(\vec{\nabla}_{31}\cdot\vec{\sigma}_1\vec{\nabla}_{31}\cdot\vec{\sigma}_3 - \left(s^2+1\right)\vec{\sigma}_1\cdot\vec{\sigma}_3\right)\\
&\times (3c_3 - c_4\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 - c_4\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3) + 4\left(s^2+1\right)(\vec{\nabla}_{12}\cdot\vec{\nabla}_{31})^2\vec{\sigma}_2\cdot\vec{\sigma}_3(3c_3 - c_4\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 - c_4\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3)\Big]\\
&\times U_1^s(x_{12})U_1^s(x_{23})U_2^s(x_{31}),
\end{aligned}
\tag{4.29}
$$

$$
\begin{aligned}
V_{\rm ring}^{(5),g_A^2} &= \frac{g_A^2 M_\pi^8}{1024\pi^4 F_\pi^6}\int_{-\infty}^{\infty} ds\Big[8c_1\vec{\nabla}_{12}\cdot\vec{\nabla}_{23}\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3 + 8c_1\vec{\nabla}_{23}\cdot\vec{\nabla}_{31}\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3 + 4c_2 s^2\vec{\nabla}_{12}\cdot\vec{\nabla}_{23}\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\\
&+ 4c_2 s^2\vec{\nabla}_{23}\cdot\vec{\nabla}_{31}\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3 + \vec{\nabla}_{12}\cdot\vec{\nabla}_{31}\Big(-4c_3\vec{\nabla}_{12}\cdot\vec{\nabla}_{23}\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3 - 4c_3\vec{\nabla}_{23}\cdot\vec{\nabla}_{31}\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\\
&+ c_4\vec{\nabla}_{12}\cdot\vec{\nabla}_{23}\times\vec{\sigma}_2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\times\boldsymbol{\tau}_3 + 4c_4\vec{\nabla}_{23}\cdot\vec{\sigma}_1\vec{\nabla}_{31}\cdot\vec{\sigma}_3(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3)\\
&+ 2c_4\left(s^2+1\right)\vec{\sigma}_2\cdot\vec{\sigma}_3(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3)\Big) + 4c_3 s^2\vec{\nabla}_{12}\cdot\vec{\nabla}_{23}\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3 + 4c_3 s^2\vec{\nabla}_{23}\cdot\vec{\nabla}_{31}\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\\
&- 2c_4\vec{\nabla}_{12}\cdot\vec{\nabla}_{23}\vec{\nabla}_{31}\cdot\vec{\sigma}_1\vec{\nabla}_{31}\cdot\vec{\sigma}_3\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 - 2c_4\vec{\nabla}_{12}\cdot\vec{\nabla}_{23}\vec{\nabla}_{31}\cdot\vec{\sigma}_1\vec{\nabla}_{31}\cdot\vec{\sigma}_3\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\\
&- 2c_4\vec{\nabla}_{12}\cdot\vec{\nabla}_{23}\vec{\nabla}_{23}\cdot\vec{\nabla}_{31}\vec{\sigma}_2\cdot\vec{\sigma}_3\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 - 2c_4\vec{\nabla}_{12}\cdot\vec{\nabla}_{23}\vec{\nabla}_{23}\cdot\vec{\nabla}_{31}\vec{\sigma}_2\cdot\vec{\sigma}_3\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3\\
&+ c_4\vec{\nabla}_{23}\cdot\vec{\nabla}_{31}\vec{\nabla}_{12}\cdot\vec{\nabla}_{23}\times\vec{\sigma}_2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\times\boldsymbol{\tau}_3 - 2c_4\left(s^2+1\right)\vec{\nabla}_{12}\cdot\vec{\sigma}_3\vec{\nabla}_{23}\cdot\vec{\sigma}_1(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3)\Big]\\
&\times U_1^s(x_{12})U_1^s(x_{23})U_1^s(x_{31}),
\end{aligned}
\tag{4.30}
$$

$$
V_{\rm ring}^{(5),g_A^0} = -\frac{M_\pi^8 s^2}{1024\pi^4 F_\pi^6}\int_{-\infty}^{\infty} ds\Big[4\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\left(2c_1 + s^2(c_2+c_3) - c_3\vec{\nabla}_{12}\cdot\vec{\nabla}_{31}\right) + c_4\vec{\nabla}_{12}\cdot\vec{\nabla}_{23}\times\vec{\sigma}_2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\times\boldsymbol{\tau}_3\Big]
$$

$$\times\ U_1^s(x_{12})U_1^s(x_{23})U_1^s(x_{31}), \tag{4.31}$$

where

$$\begin{aligned} U_1^s(x) &= \frac{4\pi}{M_\pi}\int\frac{d^3q}{(2\pi)^3}\frac{1}{q^2+\tilde{M}_\pi^2}e^{i\vec{q}\cdot\vec{x}/M_\pi} = \frac{e^{-x\sqrt{1+s^2}}}{x}, \quad \tilde{M}_\pi^2 = M_\pi^2 + l_0^2, \quad s = \frac{l_0}{M_\pi}, \\ U_2^s(x) &= 8\pi M_\pi\int\frac{d^3q}{(2\pi)^3}\frac{1}{(q^2+\tilde{M}_\pi^2)^2}e^{i\vec{q}\cdot\vec{x}/M_\pi} = \frac{1}{\sqrt{1+s^2}}e^{-x\sqrt{1+s^2}}. \end{aligned} \tag{4.32}$$

The expressions in momentum space are rather lengthy and can be found in appendix A.

V. GENERAL OPERATOR STRUCTURE OF A LOCAL THREE-NUCLEON FORCE

As already emphazized in the introduction, having derived explicit expressions for the long-range part of the 3NF at the three first orders in the chiral expansion, it is interesting to test convergence in coordinate space. One generally expects for the chiral expansion of nuclear potentials to converge at distances of the order of or larger than $r \sim M_\pi^{-1}$. In order to analyze the convergence of the chiral expansion for three-nucleon potentials in a meaningful way, we first need to define a basis in the space of isospin-spin-position or, equivalently, isospin-spin-momentum three-nucleon operators. Thus, we need to work out the most general structure of the three-nucleon force. To the best of our knowledge, this task has not been accomplished yet, see however Ref. [42], where the most general isospin structure of the 3NF is given.

Given that a general 3NF depends, in the center of mass system, on four independent momenta in addition to the spin and isospin Pauli matrices, its structure is obviously rather rich. Fortunately, even at such a high order in the chiral expansion as N^{4}LO, the most complicated part of the three-nucleon force (before antisymmetrization) is still local. For the long-range part, the only non-localities in the power counting scheme we adopt arise from the leading relativistic corrections to the 2π exchange diagrams discussed in Ref. [30]. We, therefore, restrict ourselves here to the most general structure of a local 3NF. We, furthermore, require in the following that the 3NF is invariant under parity, time-reversal and isospin transformations.

Every operator appearing in the 3NF can be written as a linear combination of spin-momentum terms multiplied with isospin structures. We remind the reader that according to the standard convention, the expressions for nuclear forces are to be understood as matrix elements with respect to momenta and operators in the spin and isospin spaces. The building blocks for the spin-momentum structures are

$$\vec{\sigma}_1,\ \vec{\sigma}_2,\ \vec{\sigma}_3,\ \vec{q}_1,\ \vec{q}_3, \tag{5.33}$$

where $\vec{\sigma}_i$ are the Pauli spin matrices of the nucleons while $\vec{q}_1$, $\vec{q}_3$ denote the two independent relative momenta.[4] These building blocks have to be contracted with the tensors δ_{ab} and ϵ_{abc} to build scalar operators. We have the following symmetry constraints:

- Parity invariance of the force allows only for spin-momentum structures which are invariant under

$$\vec{q}_1 \to -\vec{q}_1 \quad \text{and} \quad \vec{q}_3 \to -\vec{q}_3\,.$$

- Time-reversal invariance implies that only those structures contribute which are invariant under

$$\vec{\sigma}_i \to -\vec{\sigma}_i\,, \quad \vec{q}_i \to \vec{q}_i \quad \text{and} \quad \tau_i^y \to -\tau_i^y, \quad i = 1,2,3,$$

  see Eq. (2.47) of Ref. [43][5].

[4] The momentum transfer $\vec{q}_2$ can be expressed in terms of $\vec{q}_{1,3}$ via $\vec{q}_2 = -\vec{q}_1 - \vec{q}_3$.
[5] The invariance under $\tau_i^y \to -\tau_i^y$ follows directly from the invariance of the matrix element under $\langle t'|\boldsymbol{\tau}_i|t\rangle \to \langle t|\boldsymbol{\tau}_i|t'\rangle$

- Isospin conservation requires any structure to be a product of a spin-momentum operator with one of the following isospin-structures:

$$1, \quad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2, \quad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_3, \quad \boldsymbol{\tau}_2 \cdot \boldsymbol{\tau}_3 \quad \text{and} \quad \boldsymbol{\tau}_1 \cdot (\boldsymbol{\tau}_2 \times \boldsymbol{\tau}_3).$$

In addition to the symmetry constraints, we will also employ the Schouten identity

$$\delta_{i,j}\epsilon_{k,l,m} - \delta_{i,k}\epsilon_{l,m,j} + \delta_{i,l}\epsilon_{m,j,k} - \delta_{i,m}\epsilon_{j,k,l} = 0$$

to eliminate redundant structures. A general local three-nucleon force can be written in a form

$$\sum_i O_i(\vec{\sigma}_1, \vec{\sigma}_2, \vec{\sigma}_3, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3, \vec{q}_1, \vec{q}_3)\, F_i(q_1, q_3, \vec{q}_1 \cdot \vec{q}_3)\,,$$

where O_i are spin-momentum-isospin operators and the scalar structure functions F_i depend only on absolute values $|\vec{q}_1|, |\vec{q}_3|$ and on the scalar product $\vec{q}_1 \cdot \vec{q}_3$. The three-nucleon force V_{3N}^{full} in Eq. (2.2) is obviously invariant under any permutation $P \in S_3$, with S_3 denoting the permutation group:

$$\sum_i P O_i P F_i = \sum_i O_i F_i, \tag{5.34}$$

where

$$\begin{aligned} PO_i(\vec{\sigma}_1, \vec{\sigma}_2, \vec{\sigma}_3, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3, \vec{q}_1, \vec{q}_3) &= O_i(\vec{\sigma}_{P[1]}, \vec{\sigma}_{P[2]}, \vec{\sigma}_{P[3]}, \boldsymbol{\tau}_{P[1]}, \boldsymbol{\tau}_{P[2]}, \boldsymbol{\tau}_{P[3]}, \vec{q}_{P[1]}, \vec{q}_{P[3]}), \\ PF_i(q_1, q_3, \vec{q}_1 \cdot \vec{q}_3) &= F_i(q_{P[1]}, q_{P[3]}, \vec{q}_{P[1]} \cdot \vec{q}_{P[3]}). \end{aligned} \tag{5.35}$$

To understand the behavior of the structure functions under permutations of momenta it is necessary to analyze the behavior of the operators O_i under permutations. Since the operator set we consider here is complete, the permuted operator PO_i is just a linear combination of O_j's:

$$PO_i = \sum_j O_j D_{ji}(P),$$

where $D(P)$ are some invertible matrices. It is easy to see that the set of matrices D builds a representation of S_3. Indeed

$$P'PO_i = P' \sum_k O_k D_{ki}(P) = \sum_{j,k} O_j\, D_{jk}(P')\, D_{ki}(P) = \sum_j O_j D_{ji}(P'P),$$

from which immediately follows

$$D(P'P) = D(P')D(P).$$

Transformations of the structure functions F_i with respect to permutations P of the nucleon labels can now be read off from

$$\sum_i O_i F_i = \sum_i P O_i P F_i = \sum_{i,j} O_j D_{ji}(P) P F_i = \sum_i O_i \left(\sum_j D_{ij}(P) P F_j \right),$$

from which we obtain the identity

$$F_i = \sum_j D_{ij}(P) P F_j.$$

It is advantageous to choose the basis in the space of operators O_i such that the representation matrices D are block-diagonal corresponding to irreducible representations of the group S_3. There are three inequivalent irreducible representations of S_3:

- The trivial (identity) and antisymmetric $(-1)^{w(P)}$ representations with $w(P) = \pm 1$ for even/odd permutations are one dimensional.

- The third irreducible representation is two dimensional. The representation matrices can e.g. be chosen as

$$\begin{aligned}
&\mathcal{D}(()) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{D}((12)) = \tfrac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \qquad \mathcal{D}((13)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \\
&\mathcal{D}((23)) = -\tfrac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \ \mathcal{D}((123)) = -\tfrac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \ \mathcal{D}((132)) = -\tfrac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},
\end{aligned} \tag{5.36}$$

where we used the cyclic notation for permutations:

$$\begin{aligned}
()[1] &= 1, \quad ()[2] = 2, \quad ()[3] = 3, \\
(12)[1] &= 2, \quad (12)[2] = 1, \quad (12)[3] = 3, \\
(13)[1] &= 3, \quad (13)[2] = 2, \quad (13)[3] = 1, \\
(23)[1] &= 1, \quad (23)[2] = 3, \quad (23)[3] = 2, \\
(123)[1] &= 2, \quad (123)[2] = 3, \quad (123)[3] = 1, \\
(132)[1] &= 3, \quad (132)[2] = 1, \quad (132)[3] = 2.
\end{aligned} \tag{5.37}$$

To construct the operators O_i for which $D(P)$ is block-diagonal, we introduce the symmetrizing and antisymmetrizing functions:

$$S(O) = \frac{1}{6}\sum_{P\in S_3} PO, \quad A(O) = \frac{1}{6}\sum_{P\in S_3} (-1)^{w(P)} PO.$$

It is obvious that

$$PS(O) = S(O) \quad \text{and} \quad PA(O) = (-1)^{w(P)} A(O)$$

for all $P \in S_3$ such that $S(O)$ and $A(O)$ transform under one-dimensional representations. To construct operators which transform under two-dimensional irreducible representation we introduce the functions

$$G_{ij}(O) = \frac{1}{3}\sum_{P\in S_3} \mathcal{D}_{ij}(P)PO, \quad i, j = 1, 2.$$

It is easy to verify that the resulting operators $G_{ij}(O)$ indeed transform under two-dimensional irreducible representation:

$$PG_{ij}(O) = \frac{1}{3}\sum_{Q\in S_3} \mathcal{D}_{ij}(Q)PQO = \frac{1}{3}\sum_{Q\in S_3} \mathcal{D}_{ij}(P^{-1}Q)QO = \sum_k G_{kj}(O)\mathcal{D}_{ki}(P).$$

With all the symmetry constraints introduced above, we found that the most general structure of a local 3NF can be written in terms of 89 operators $O_1, \dots, O_{89}$, which transform with respect to permutations according to irreducible representations of S_3. These 89 operators can be generated from a set of 22 independent operators $\mathcal{G}_1, \dots, \mathcal{G}_{22}$ using the functions S, A and G_{ij} defined above. The explicit form of generating operators $\mathcal{G}_1, \dots, \mathcal{G}_{22}$ we adopt in this work and their relation to the generated operators $O_1, \dots, O_{89}$ are given in Table I. The complete expression for a local three-nucleon force in our notation can be written in the symmetric form

$$V_{\rm 3N}^{\rm full} = \sum_{i=1}^{89} O_i(\vec{\sigma}_1, \vec{\sigma}_2, \vec{\sigma}_3, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3, \vec{q}_1, \vec{q}_3) F_i(q_1, q_3, \vec{q}_1 \cdot \vec{q}_3). \tag{5.38}$$

In this representation, the structure functions F_i have simple transformation properties with respect to permutations[6]. An alternative way to express the three-nucleon force is given by

$$V_{\rm 3N}^{\rm full} = \sum_{i=1}^{22} \mathcal{G}_i(\vec{\sigma}_1, \vec{\sigma}_2, \vec{\sigma}_3, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3, \vec{q}_1, \vec{q}_3) \mathcal{F}_i(q_1, q_3, \vec{q}_1 \cdot \vec{q}_3) + 5\,\text{permutations}. \tag{5.39}$$

[6] The only exception is F_{17} which mixes different contributions from other structure functions. This is due to the use of the Schouten identities.

Generators $\mathcal{G}$ of 89 independent operators	S	A	G_{12}	G_{22}	G_{11}	G_{21}
$\mathcal{G}_1 = 1$	O_1	0	0	0	0	0
$\mathcal{G}_2 = \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_3$	O_2	0	O_3	O_4	0	0
$\mathcal{G}_3 = \vec{\sigma}_1 \cdot \vec{\sigma}_3$	O_5	0	O_6	O_7	0	0
$\mathcal{G}_4 = \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_3 \vec{\sigma}_1 \cdot \vec{\sigma}_3$	O_8	0	O_9	O_{10}	0	0
$\mathcal{G}_5 = \boldsymbol{\tau}_2 \cdot \boldsymbol{\tau}_3 \vec{\sigma}_1 \cdot \vec{\sigma}_2$	O_{11}	O_{12}	O_{13}	O_{14}	O_{15}	O_{16}
$\mathcal{G}_6 = \boldsymbol{\tau}_1 \cdot (\boldsymbol{\tau}_2 \times \boldsymbol{\tau}_3) \vec{\sigma}_1 \cdot (\vec{\sigma}_2 \times \vec{\sigma}_3)$	O_{17}	0	0	0	0	0
$\mathcal{G}_7 = \boldsymbol{\tau}_1 \cdot (\boldsymbol{\tau}_2 \times \boldsymbol{\tau}_3) \vec{\sigma}_2 \cdot (\vec{q}_1 \times \vec{q}_3)$	O_{18}	0	O_{19}	O_{20}	0	0
$\mathcal{G}_8 = \vec{q}_1 \cdot \vec{\sigma}_1 \vec{q}_1 \cdot \vec{\sigma}_3$	O_{21}	O_{22}	O_{23}	O_{24}	O_{25}	O_{26}
$\mathcal{G}_9 = \vec{q}_1 \cdot \vec{\sigma}_3 \vec{q}_3 \cdot \vec{\sigma}_1$	O_{27}	0	O_{28}	O_{29}	0	0
$\mathcal{G}_{10} = \vec{q}_1 \cdot \vec{\sigma}_1 \vec{q}_3 \cdot \vec{\sigma}_3$	O_{30}	0	O_{31}	O_{32}	0	0
$\mathcal{G}_{11} = \boldsymbol{\tau}_2 \cdot \boldsymbol{\tau}_3 \vec{q}_1 \cdot \vec{\sigma}_1 \vec{q}_1 \cdot \vec{\sigma}_2$	O_{33}	O_{34}	O_{35}	O_{36}	O_{37}	O_{38}
$\mathcal{G}_{12} = \boldsymbol{\tau}_2 \cdot \boldsymbol{\tau}_3 \vec{q}_1 \cdot \vec{\sigma}_1 \vec{q}_3 \cdot \vec{\sigma}_2$	O_{39}	O_{40}	O_{41}	O_{42}	O_{43}	O_{44}
$\mathcal{G}_{13} = \boldsymbol{\tau}_2 \cdot \boldsymbol{\tau}_3 \vec{q}_3 \cdot \vec{\sigma}_1 \vec{q}_1 \cdot \vec{\sigma}_2$	O_{45}	O_{46}	O_{47}	O_{48}	O_{49}	O_{50}
$\mathcal{G}_{14} = \boldsymbol{\tau}_2 \cdot \boldsymbol{\tau}_3 \vec{q}_3 \cdot \vec{\sigma}_1 \vec{q}_3 \cdot \vec{\sigma}_2$	O_{51}	O_{52}	O_{53}	O_{54}	O_{55}	O_{56}
$\mathcal{G}_{15} = \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_3 \vec{q}_2 \cdot \vec{\sigma}_1 \vec{q}_2 \cdot \vec{\sigma}_3$	O_{57}	0	O_{58}	O_{59}	0	0
$\mathcal{G}_{16} = \boldsymbol{\tau}_2 \cdot \boldsymbol{\tau}_3 \vec{q}_3 \cdot \vec{\sigma}_2 \vec{q}_3 \cdot \vec{\sigma}_3$	O_{60}	O_{61}	O_{62}	O_{63}	O_{64}	O_{65}
$\mathcal{G}_{17} = \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_3 \vec{q}_1 \cdot \vec{\sigma}_1 \vec{q}_3 \cdot \vec{\sigma}_3$	O_{66}	0	O_{67}	O_{68}	0	0
$\mathcal{G}_{18} = \boldsymbol{\tau}_1 \cdot (\boldsymbol{\tau}_2 \times \boldsymbol{\tau}_3) \vec{\sigma}_1 \cdot \vec{\sigma}_3 \vec{\sigma}_2 \cdot (\vec{q}_1 \times \vec{q}_3)$	O_{69}	0	O_{70}	O_{71}	0	0
$\mathcal{G}_{19} = \boldsymbol{\tau}_1 \cdot (\boldsymbol{\tau}_2 \times \boldsymbol{\tau}_3) \vec{\sigma}_3 \cdot \vec{q}_1 \vec{q}_1 \cdot (\vec{\sigma}_1 \times \vec{\sigma}_2)$	O_{72}	O_{73}	O_{74}	O_{75}	O_{76}	O_{77}
$\mathcal{G}_{20} = \boldsymbol{\tau}_1 \cdot (\boldsymbol{\tau}_2 \times \boldsymbol{\tau}_3) \vec{\sigma}_1 \cdot \vec{q}_1 \vec{\sigma}_2 \cdot \vec{q}_1 \vec{\sigma}_3 \cdot (\vec{q}_1 \times \vec{q}_3)$	O_{78}	O_{79}	O_{80}	O_{81}	O_{82}	O_{83}
$\mathcal{G}_{21} = \boldsymbol{\tau}_1 \cdot (\boldsymbol{\tau}_2 \times \boldsymbol{\tau}_3) \vec{\sigma}_1 \cdot \vec{q}_2 \vec{\sigma}_3 \cdot \vec{q}_2 \vec{\sigma}_2 \cdot (\vec{q}_1 \times \vec{q}_3)$	O_{84}	0	O_{85}	O_{86}	0	0
$\mathcal{G}_{22} = \boldsymbol{\tau}_1 \cdot (\boldsymbol{\tau}_2 \times \boldsymbol{\tau}_3) \vec{\sigma}_1 \cdot \vec{q}_1 \vec{\sigma}_3 \cdot \vec{q}_3 \vec{\sigma}_2 \cdot (\vec{q}_1 \times \vec{q}_3)$	O_{87}	0	O_{88}	O_{89}	0	0

TABLE I: The set of 22 generating operators $\mathcal{G}_i$ and their relation to 89 independent operators $O_1, \ldots, O_{89}$ which parametrize the most general structure of a local 3NF. The operators O_i are generated by application of one of the 6 functions $S, A, G_{11}, G_{12}, G_{21}, G_{22}$ on the corresponding operator $\mathcal{G}_j$. The 22 operators are constructed to be either totally symmetric, symmetric under $1 \leftrightarrow 3$ or unsymmetric.

It is easy to see that the two representations (5.39) and (5.38) are equivalent. While Eq. (5.39) can obviously be brought into the form of Eq. (5.38) we now show that the converse is also true. Eq. (5.38) can be rewritten in the form

$$
\begin{aligned}
V_{3\mathrm{N}}^{\mathrm{full}} &= \sum_{i=1}^{22} \left\{ S(\mathcal{G}_i) M_i + A(\mathcal{G}_i) N_i + \sum_{j,k=1}^{2} G_{jk}(\mathcal{G}_i) L^i_{jk} \right\} \\
&= \sum_{P \in S_3} \sum_{i=1}^{22} P(\mathcal{G}_i) \left\{ \frac{1}{6} M_i + \frac{1}{6} (-1)^{w(P)} N_i + \sum_{j,k=1}^{2} \frac{1}{3} \mathcal{D}_{jk}(P) L^i_{jk} \right\},
\end{aligned} \tag{5.40}
$$

where M_i, N_i, L^i_{jk} are some of the structure functions $F_l (l = 1, \ldots, 89)$ from Eq. (5.38). From the symmetry property (5.34) we get

$$
\begin{aligned}
V_{3\mathrm{N}}^{\mathrm{full}} &= \frac{1}{6} \sum_{P' \in S_3} P'(V_{3\mathrm{N}}^{\mathrm{full}}) \\
&= \frac{1}{6} \sum_{P', P \in S_3} \sum_{i=1}^{22} P' P(\mathcal{G}_i) \left\{ \frac{1}{6} P'(M_i) + \frac{1}{6} (-1)^{w(P)} P'(N_i) + \sum_{j,k=1}^{2} \frac{1}{3} \mathcal{D}_{jk}(P) P'(L^i_{jk}) \right\} \\
&= \sum_{P'' \in S_3} P'' \sum_{i=1}^{22} \mathcal{G}_i \sum_{P \in S_3} \left\{ \frac{1}{36} P^{-1}(M_i) + \frac{1}{36} (-1)^{w(P)} P^{-1}(N_i) + \sum_{j,k=1}^{2} \frac{1}{18} \mathcal{D}_{jk}(P) P^{-1}(L^i_{jk}) \right\},
\end{aligned} \tag{5.41}
$$

where we made a change of variable $P'' = P'P$ in the last line. This equation has the form of Eq. (5.39) with

$$\mathcal{F}_i \;:=\; \sum_{P\in S_3} \left\{ \frac{1}{36} P^{-1}(M_i) + \frac{1}{36}(-1)^{w(P)} P^{-1}(N_i) + \sum_{j,k=1}^{2} \frac{1}{18} \mathcal{D}_{jk}(P) P^{-1}(L^i_{jk}) \right\} . \tag{5.42}$$

VI. CHIRAL EXPANSION OF THE LONG-RANGE TAIL OF THE 3NF

With these preparations we are now in the position to address the convergence of the chiral expansion for the long-range tail of the 3NF. It is clear that all arguments of the previous section can also be applied to operators in coordinate space. Here and in what follows, we use the following basis of 22 operators:

$$\begin{aligned}
\tilde{\mathcal{G}}_1 &= 1\,,\\
\tilde{\mathcal{G}}_2 &= \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3\,,\\
\tilde{\mathcal{G}}_3 &= \vec\sigma_1\cdot\vec\sigma_3\,,\\
\tilde{\mathcal{G}}_4 &= \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3\,\vec\sigma_1\cdot\vec\sigma_3\,,\\
\tilde{\mathcal{G}}_5 &= \boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\,\vec\sigma_1\cdot\vec\sigma_2\,,\\
\tilde{\mathcal{G}}_6 &= \boldsymbol{\tau}_1\cdot(\boldsymbol{\tau}_2\times\boldsymbol{\tau}_3)\,\vec\sigma_1\cdot(\vec\sigma_2\times\vec\sigma_3)\,,\\
\tilde{\mathcal{G}}_7 &= \boldsymbol{\tau}_1\cdot(\boldsymbol{\tau}_2\times\boldsymbol{\tau}_3)\,\vec\sigma_2\cdot(\hat r_{12}\times\hat r_{23})\,,\\
\tilde{\mathcal{G}}_8 &= \hat r_{23}\cdot\vec\sigma_1\,\hat r_{23}\cdot\vec\sigma_3\,,\\
\tilde{\mathcal{G}}_9 &= \hat r_{23}\cdot\vec\sigma_3\,\hat r_{12}\cdot\vec\sigma_1\,,\\
\tilde{\mathcal{G}}_{10} &= \hat r_{23}\cdot\vec\sigma_1\,\hat r_{12}\cdot\vec\sigma_3\,,\\
\tilde{\mathcal{G}}_{11} &= \boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\,\hat r_{23}\cdot\vec\sigma_1\,\hat r_{23}\cdot\vec\sigma_2\,,\\
\tilde{\mathcal{G}}_{12} &= \boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\,\hat r_{23}\cdot\vec\sigma_1\,\hat r_{12}\cdot\vec\sigma_2\,,\\
\tilde{\mathcal{G}}_{13} &= \boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\,\hat r_{12}\cdot\vec\sigma_1\,\hat r_{23}\cdot\vec\sigma_2\,,\\
\tilde{\mathcal{G}}_{14} &= \boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\,\hat r_{12}\cdot\vec\sigma_1\,\hat r_{12}\cdot\vec\sigma_2\,,\\
\tilde{\mathcal{G}}_{15} &= \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3\,\hat r_{13}\cdot\vec\sigma_1\,\hat r_{13}\cdot\vec\sigma_3\,,\\
\tilde{\mathcal{G}}_{16} &= \boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\,\hat r_{12}\cdot\vec\sigma_2\,\hat r_{12}\cdot\vec\sigma_3\,,\\
\tilde{\mathcal{G}}_{17} &= \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3\,\hat r_{23}\cdot\vec\sigma_1\,\hat r_{12}\cdot\vec\sigma_3\,,\\
\tilde{\mathcal{G}}_{18} &= \boldsymbol{\tau}_1\cdot(\boldsymbol{\tau}_2\times\boldsymbol{\tau}_3)\,\vec\sigma_1\cdot\vec\sigma_3\,\vec\sigma_2\cdot(\hat r_{12}\times\hat r_{23})\,,\\
\tilde{\mathcal{G}}_{19} &= \boldsymbol{\tau}_1\cdot(\boldsymbol{\tau}_2\times\boldsymbol{\tau}_3)\,\vec\sigma_3\cdot\hat r_{23}\,\hat r_{23}\cdot(\vec\sigma_1\times\vec\sigma_2)\,,\\
\tilde{\mathcal{G}}_{20} &= \boldsymbol{\tau}_1\cdot(\boldsymbol{\tau}_2\times\boldsymbol{\tau}_3)\,\vec\sigma_1\cdot\hat r_{23}\,\vec\sigma_2\cdot\hat r_{23}\,\vec\sigma_3\cdot(\hat r_{12}\times\hat r_{23})\,,\\
\tilde{\mathcal{G}}_{21} &= \boldsymbol{\tau}_1\cdot(\boldsymbol{\tau}_2\times\boldsymbol{\tau}_3)\,\vec\sigma_1\cdot\hat r_{13}\,\vec\sigma_3\cdot\hat r_{13}\,\vec\sigma_2\cdot(\hat r_{12}\times\hat r_{23})\,,\\
\tilde{\mathcal{G}}_{22} &= \boldsymbol{\tau}_1\cdot(\boldsymbol{\tau}_2\times\boldsymbol{\tau}_3)\,\vec\sigma_1\cdot\hat r_{23}\,\vec\sigma_3\cdot\hat r_{12}\,\vec\sigma_2\cdot(\hat r_{12}\times\hat r_{23})\,,
\end{aligned} \tag{6.43}$$

where $\hat r_{ij} \equiv \vec r_{ij}/|\vec r_{ij|}$ and $\vec r_{ij} = \vec r_i - \vec r_j$ denotes the position of nucleon i with respect to nucleon j. The 3NF is a linear combination of the operators $\tilde{\mathcal{G}}_i$ with the coefficients given by scalar functions $\mathcal{F}_i(r_{12}, r_{23}, r_{31})$. These functions have the dimension of energy and can be interpreted as the potential energy between three static nucleons projected onto the corresponding operator. The profile functions $\mathcal{F}_i$ receive contributions from the long-range and the intermediate-range 3NF topologies and are predicted (at long distances) in terms of the chiral expansion. In order to explore the convergence, we plot these functions for the equilateral triangle configuration of the nucleons given by the condition

$$r_{12} = r_{23} = r_{31} = r\,. \tag{6.44}$$

Restricting ourselves to this particular configuration allows us to stay with simple one-dimensional plots. We emphasize, however, that the conclusions about the convergence of the chiral expansion for the 3NF drawn in this section apply to this particular configuration. We begin with the longest-range 2π exchange topology. Projecting the coordinate-space expressions given in section II onto the operators in Eq. (6.43) and evaluating the three-dimensional integrals in Eqs. (2.8) and (2.13) numerically we compute the corresponding contributions to the profile functions $\mathcal{F}^{(3)}(r)$, $\mathcal{F}^{(4)}(r)$ and $\mathcal{F}^{(5)}(r)$ at N^{2}LO, N^{3}LO and N^{4}LO, respectively. Our results for the 3NF profile functions

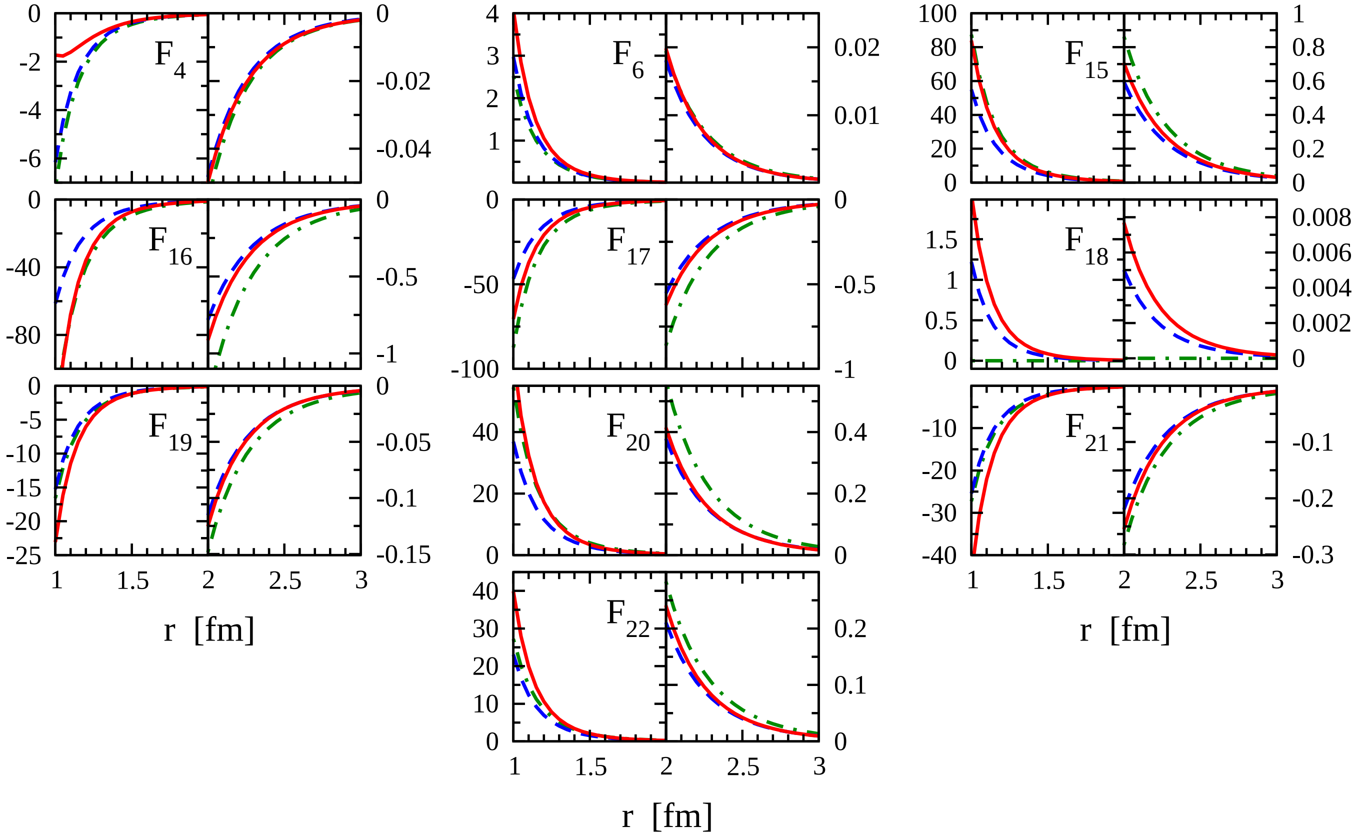

FIG. 4: Chiral expansion of the profile functions $\mathcal{F}_i(r)$ in MeV generated by the two-pion exchange 3NF topology up to N^{4}LO (in the equilateral triangle configuration). Dashed-dotted, dashed and solid lines correspond to $\mathcal{F}_i^{(3)}$, $\mathcal{F}_i^{(3)} + \mathcal{F}_i^{(4)}$ and $\mathcal{F}_i^{(3)} + \mathcal{F}_i^{(4)} + \mathcal{F}_i^{(5)}$, respectively.

generated by the 2π exchange diagrams are visualized in Fig. 4. Here and in what follows, we use the values of the low-energy constants corresponding to the order-Q^4 KH fit to the pion-nucleon partial wave analysis of our work [40]. In particular, we employ the following values of c_i (all in units of GeV^{-1}):

$$c_1 = -0.75, \quad c_2 = 3.49, \quad c_3 = -4.77, \quad c_4 = 3.34\,. \tag{6.45}$$

The results for the functions $\mathcal{F}_i(r)$ plotted in Fig. 4 resemble the findings of our work [40], where a good convergence of the chiral expansion in momentum space was observed by looking at the functions $\mathcal{A}(q_2)$ and $\mathcal{B}(q_2)$ for low values of the momentum transfer. While there are large corrections at N^{4}LO to some of the profile functions and, especially, to $\mathcal{F}_4(r)$ at short distances of the order of $r \sim 1$ fm, we observe a very good convergence at long distances of the order of $r \gtrsim 2$ fm. At such large distances, the N^{4}LO results appear to be very close to N^{3}LO ones. As already pointed out in the introduction, fast convergence of the longest-range 3NF is not surprising given that effects of the Δ-isobar are, to a large extent, accounted for already in the leading contributions $\mathcal{F}_i^{(3)}(r)$ to this topology through resonance saturation of the LECs $c_{3,4}$. We further emphasize that the operator structure of the 2π exchange topology is fairly restricted: only 10 out of 22 functions $\mathcal{F}_i(r)$ get non-vanishing contributions. Notice that the larger number of nonvanishing functions $\mathcal{F}_i$ in coordinate space compared to momentum space has to be expected due to the appearance of gradients when performing the Fourier transform. In contrast to the momentum space representation, the number of nonvanishing structures in the coordinate space representation of a 3NF is not representative for estimating the number of affected nucleon-deuteron polarization observables at a fixed kinematics.

It is instructive to compare the strength of the three- and two-nucleon potentials. While the long-range three-nucleon potentials are considerably weaker than the two-nucleon potentials, they are still not negligible. For example, the isovector-tensor and isoscalar central nucleon-nucleon potentials governed by one-pion exchange and (subleading)

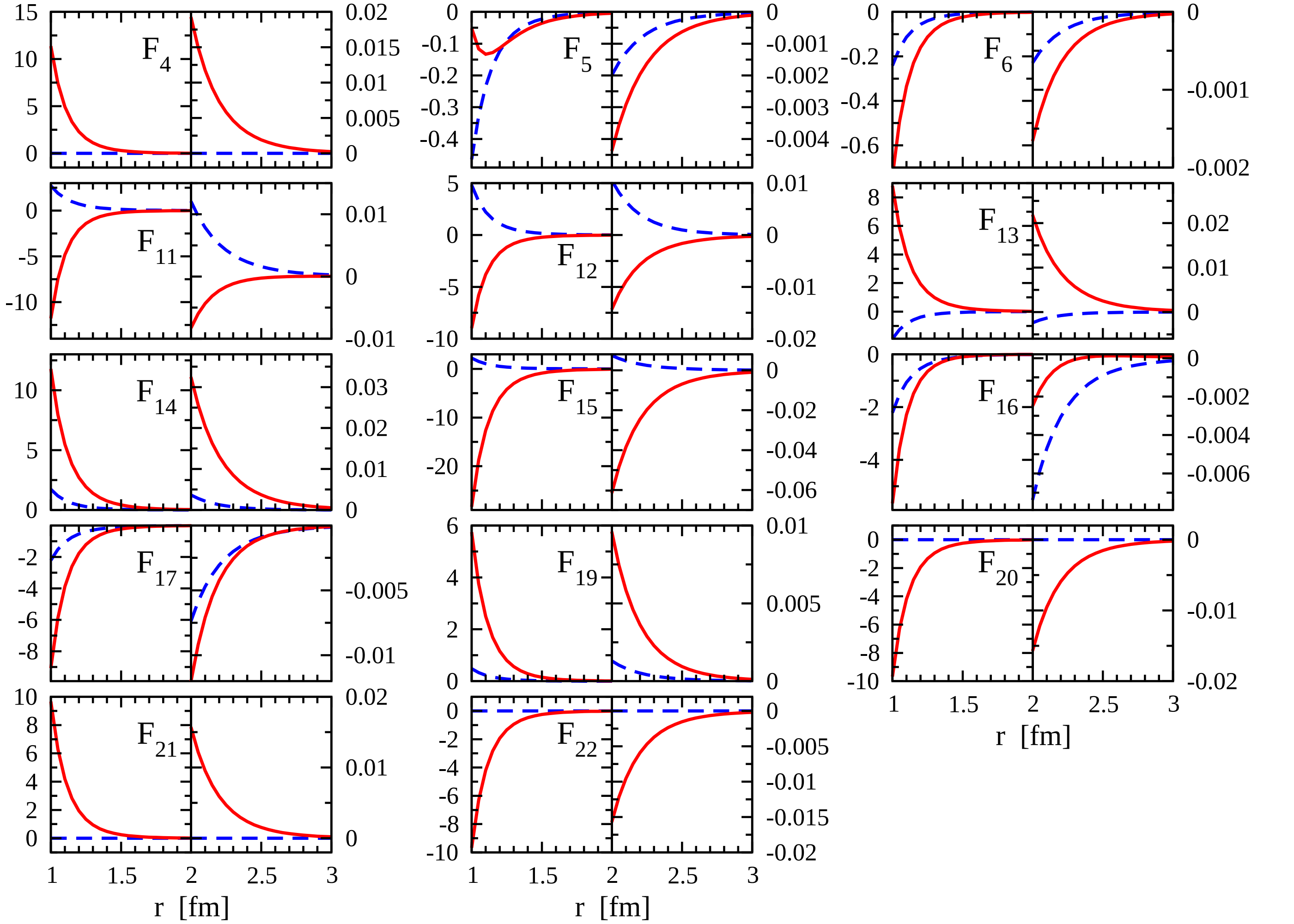

FIG. 5: Chiral expansion of the profile functions $\mathcal{F}_i(r)$ in MeV generated by the two-pion-one-pion exchange 3NF topology up to N^{4}LO (in the equilateral triangle configuration). Dashed and solid lines correspond to $\mathcal{F}_i^{(4)}$ and $\mathcal{F}_i^{(4)} + \mathcal{F}_i^{(5)}$, respectively.

two-pion exchange, respectively, have the strength of the order of $3\ldots4$ MeV at distances $r \sim 2$ fm [15]. These are the strongest two-nucleon forces at large distances. The strongest three-nucleon potentials $\mathcal{F}_{15}(r), \mathcal{F}_{16}(r)$ and $\mathcal{F}_{17}(r)$ reach at such distances the strength of $\sim 0.7\ldots1$ MeV. We remind the reader that nuclear potentials become scheme dependent at short distances below $r \sim 1\ldots1.5$ fm, where the contributions of short-range topologies start playing important role. An estimation of this intrinsic scheme dependence for nucleon-nucleon potentials is provided in Fig. 3 of Ref. [15].

The convergence of the chiral expansion for the two-pion-one-pion exchange and ring topologies is, as expected, much worse, see Figs. 5 and 6. In nearly all cases, the subleading contributions at N^{4}LO dominate over the nominally leading ones at N^{3}LO even at large distances. This is analogous to the pattern observed for the two-pion exchange two-nucleon potential. In that case, the strong dominance of the subleading terms appears because of several reasons including the large numerical coefficients, an enhancement by one power of π as compared to the standard chiral power counting which is characteristic to the triangle diagrams, see also Ref. [44], as well as the large numerical values of the LECs $c_{3,4}$ from the subleading pion-nucleon effective Lagrangian which are governed by the Δ isobar. In the case of the 3NF 2π-1π exchange and ring potentials the situation is less dramatic. In particular, the enhancement by a power of π actually affects the leading contributions at N^{3}LO which involve the loop function $A(q_2)$. Still, the corrections at N^{4}LO are large which can presumably be attributed to the large numerical values of the LECs c_i. One should, however, emphasize that the potentials generated by the 2π-1π exchange and ring diagrams have a considerably

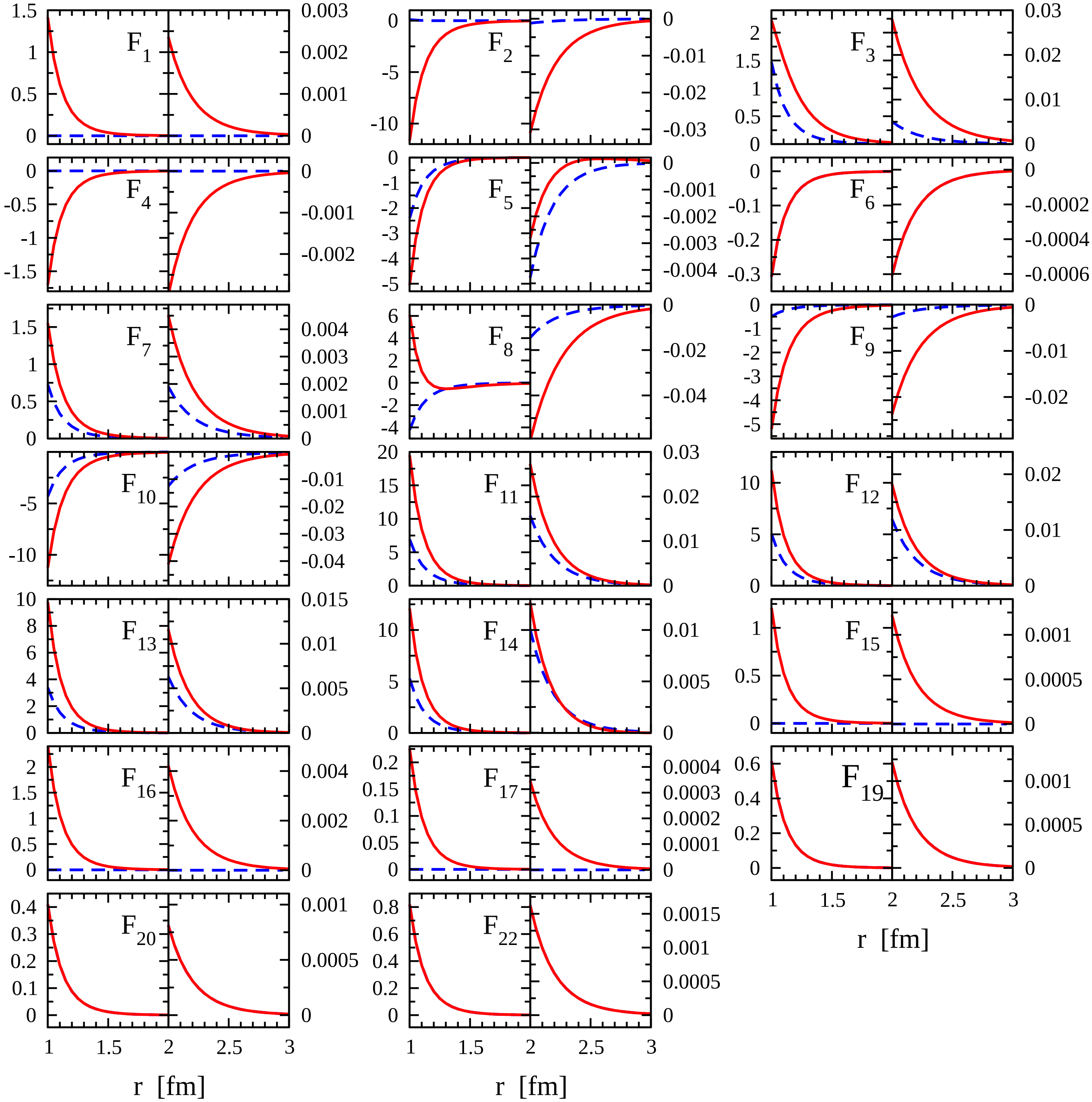

FIG. 6: Chiral expansion of the profile functions $\mathcal{F}_i(r)$ in MeV generated by the ring 3NF topology up to N^{4}LO (in the equilateral triangle configuration). Dashed and solid lines correspond to $\mathcal{F}_i^{(4)}$ and $\mathcal{F}_i^{(4)} + \mathcal{F}_i^{(5)}$, respectively.

shorter range as compared to the 2π exchange ones and only reach at most ~ 50 keV at distances of the order of $r \sim 2$ fm. It is, therefore, not clear whether the lack of convergence will have any significant phenomenological impact. Clearly, to answer this question one needs to carry out complete calculations of few- and many-nucleon observables. This work is in progress. Last but not least, we emphasize that especially the ring topology generates a very rich structure in the 3NF and gives rise to 20 out of 22 profile functions.

It is also instructive to compare the 3NF potentials corresponding to the individual topologies with each other. Such a comparison is given in Fig. 7, where we restrict ourselves to N^{4}LO, i.e. we only show $\mathcal{F}_i^{(3)} + \mathcal{F}_i^{(4)} + \mathcal{F}_i^{(5)}$. We observe that the 2π-1π exchange and ring potentials are of a comparable size. However, in all cases where the longest-range 2π exchange topology contribute, it clearly dominates at $r \gtrsim 2$ fm over the two other topologies. At shorter distances of the order of $r \sim 1$ fm the impact of the 2π-1π exchange and ring terms becomes more significant with, e.g. $|\mathcal{F}_{11,15}(1 \text{ fm})| \sim 20$ MeV to be compared with the strongest 2π exchange potentials $|\mathcal{F}_{15,16,17}(1 \text{ fm})| \sim 100$ MeV. As pointed out before, it is difficult to draw conclusions on the phenomenological importance of the new structures based on this comparison alone since one generally expects that (scheme-dependent) short-range contributions to the 3NF not considered in the present work would become significant at $r \lesssim 1$ fm.

Last but not least, Fig. 8 shows the chiral expansion of the complete results for the functions $\mathcal{F}_i(r)$ emerging from adding the contributions from all three topologies together. The interpretation follows directly from the above discussion. At long distances of the order of $r \gtrsim 2$ fm dominated by the 2π exchange one observes a good convergence for all cases where the potentials are numerically sizable. On the other hand, those profile functions which are not affected by the 2π exchange are typically dominated by the N^{4}LO contributions which might be still not converged at this order in the chiral expansion. The corresponding potentials are, however, rather weak. At shorter distances $r \sim 1 \ldots 2$ fm, the 2π-1π exchange and ring contributions start becoming more important relative to the 2π exchange terms. One again observes the dominance of the N^{4}LO contributions which supports the assumption about the important role played by Δ excitations, whose effects are partially taken into account at N^{4}LO through resonance saturation of the LECs c_2, c_3 and c_4.

VII. SUMMARY AND OUTLOOK

In this paper, we have worked out and analyzed in detail the intermediate-range contributions to the three-nucleon force at N^{4}LO, which emerge from the 2π-1π exchange and ring topologies. We used here the heavy-baryon formulation of chiral EFT with pions and nucleons being the only explicit degrees of freedom. The pertinent results of our study can be summarized as follows.

- We worked out the coordinate-space representation of the N^{4}LO corrections to the 2π exchange 3NF calculated in momentum space in Ref. [40].
- We derived the N^{4}LO contributions to the intermediate-range 2π-1π exchange and ring topologies. These represent the first corrections to the leading results which appear at N^{3}LO and have been worked out in Ref. [28]. We provide explicit analytical expressions in both momentum and coordinate spaces. The obtained corrections do not involve any unknown low-energy constants.
- We have demonstrated that the most general structure of an isospin-invariant local three-nucleon force involves 89 independent isospin-spin-momentum operators. We proposed a set of 22 linearly-independent operators which can serve as a basis and gives rise to all 89 structures in the 3NF upon making permutations of nucleon labels. We also discussed the properties of the corresponding scalar structure functions $\mathcal{F}_{1\ldots22}$ parametrizing the 3NF with respect to permutations.
- Finally, using the above mentioned operator basis, we addressed the convergence of the chiral expansion for the long-range tail of the 3NF in the equilateral triangle configuration with $r_{12} = r_{23} = r_{31} = r$ by comparing our predictions for the potentials $\mathcal{F}_i$ at different orders in the chiral expansion. Consistently with the momentum-space results of Ref. [40], we observe a good convergence for the longest-range 2π exchange topology which clearly dominates the 3NF at distances of the order $r \gtrsim 2$ fm. The intermediate-range 2π-1π exchange and ring diagrams provide sizable corrections to $\mathcal{F}_i$ at $r \sim 1$ fm and also contribute to those 12 profile functions which vanish for the 2π exchange. As expected, we found that N^{4}LO corrections to the intermediate-range topologies are numerically large and in most cases dominate over the nominally leading N^{3}LO terms. This can be traced back to the role played by the $\Delta(1232)$ isobar whose excitations provide an important 3NF mechanism. In the standard, delta-less formulation of chiral EFT we employ here, effects of the Δ isobar are not incorporated in N^{3}LO contributions to the 3NF. For the intermediate-range topologies we are primarily interested in here, first effects of the Δ appear at N^{4}LO through resonance saturation of the LECs c_2, c_3 and c_4 which accompany the subleading pion-nucleon vertices in the effective Lagrangian. The importance of the Δ isobar is reflected in the numerically large values of these LECs which are responsible for large N^{4}LO corrections we observe.

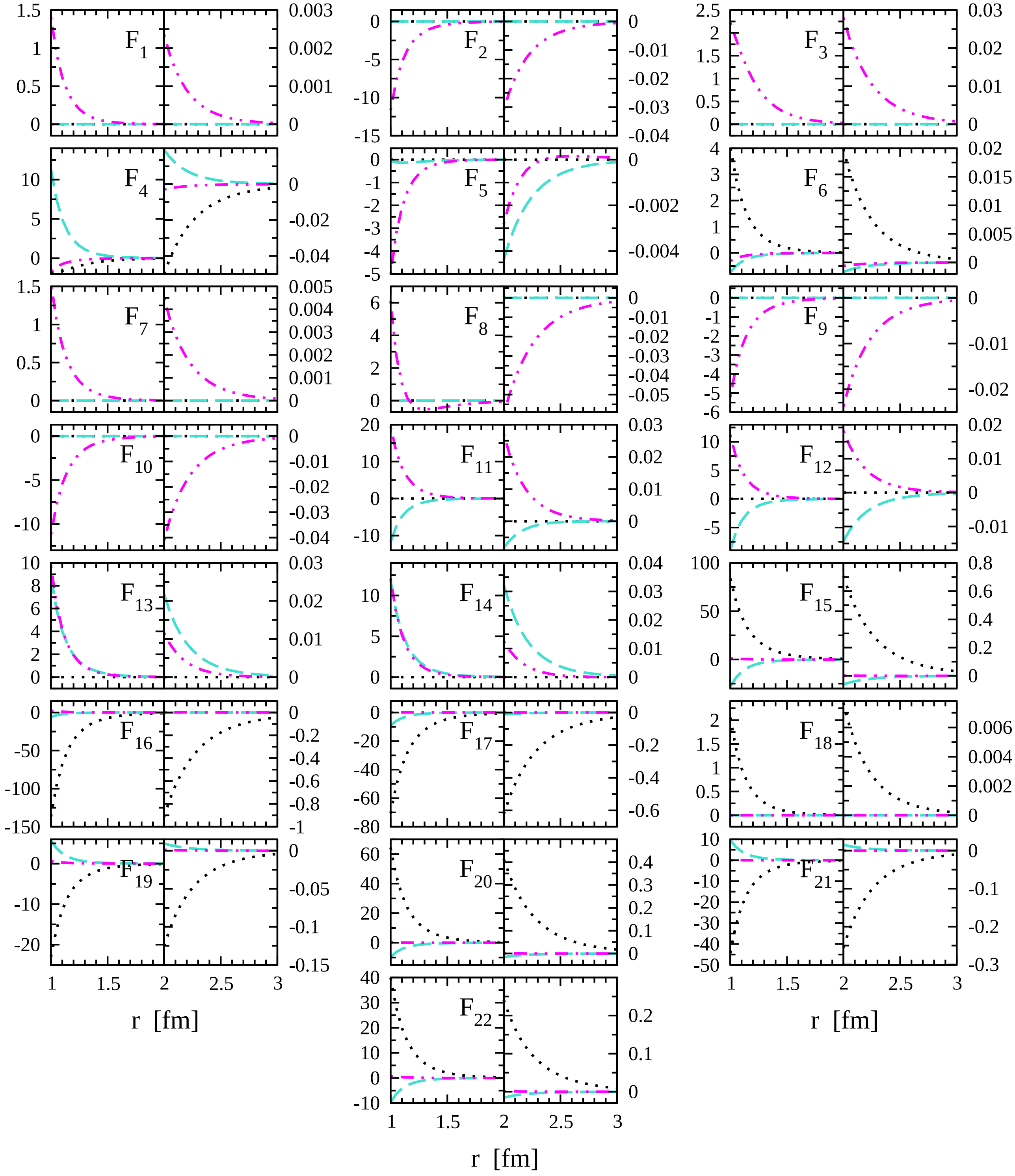

FIG. 7: Individual contributions of the two-pion exchange (dotted lines), two-pion-one-pion exchange (long-dashed lines) and ring (dashed-double-dotted lines) topologies to the profile functions $\mathcal{F}_i(r)$ in MeV at N^{4}LO in the equilateral triangle configuration.

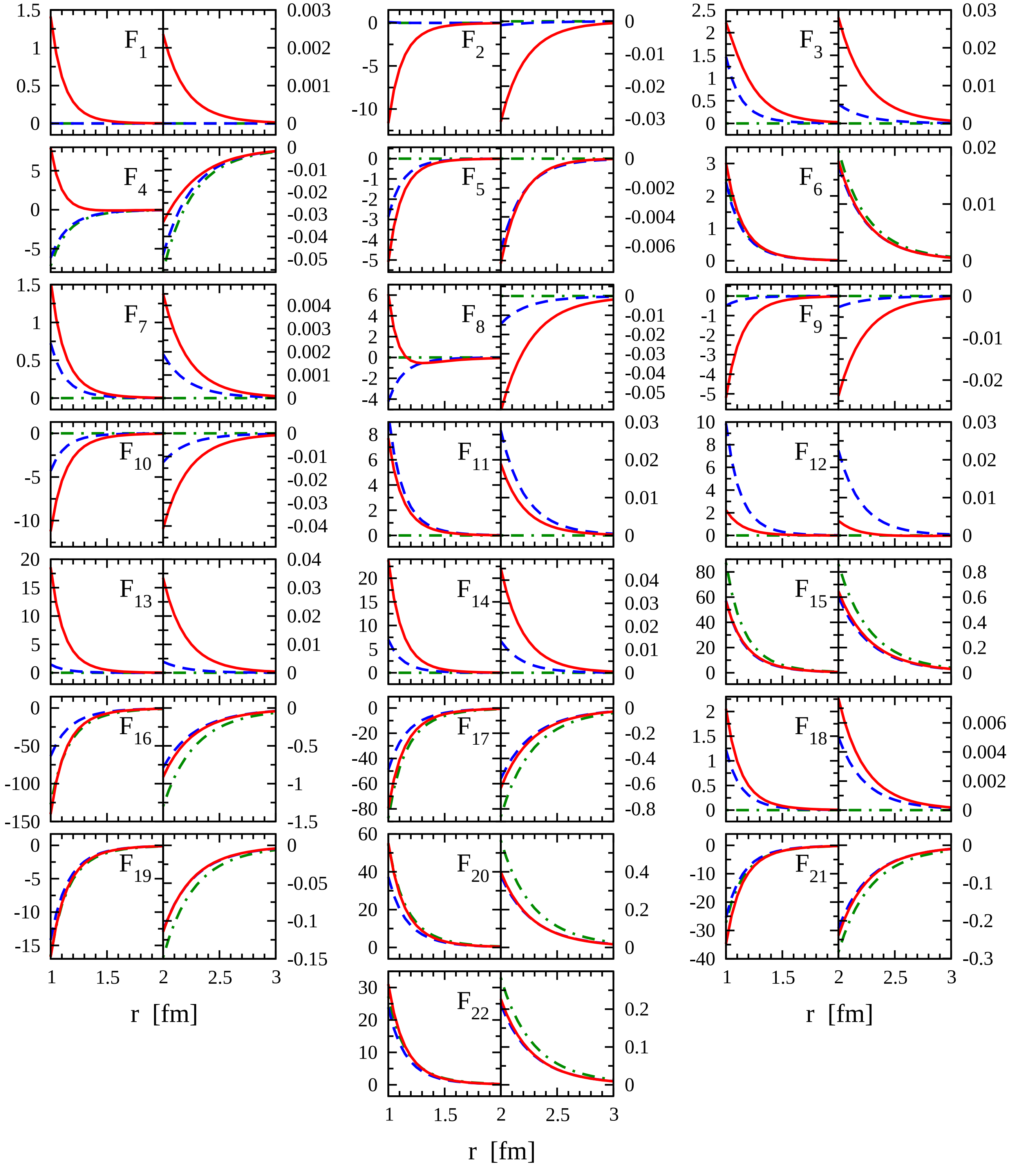

FIG. 8: Chiral expansion of the profile functions $\mathcal{F}_i(r)$ in MeV emerging from all long-range 3NF topologies up to N^{4}LO (in the equilateral triangle configuration). Dashed-dotted, dashed and solid lines correspond to $\mathcal{F}_i^{(3)}$, $\mathcal{F}_i^{(3)}+\mathcal{F}_i^{(4)}$ and $\mathcal{F}_i^{(3)}+\mathcal{F}_i^{(4)}+\mathcal{F}_i^{(5)}$, respectively.

The results of our work provide important step towards precise, quantitative theoretical description of the 3NF in the framework of chiral EFT. The long-range part of the 3NF is governed by exchange of pions, the Goldstone bosons of the spontaneously broken chiral symmetry of QCD, and can be rigorously calculated in the framework of chiral EFT. It is expected to affect the energy dependence of the nucleon-deuteron scattering amplitude at low energies and might be responsible for the long-standing puzzles such as e.g. the A_y puzzle in elastic three- and four-nucleon scattering [3]. Although the resulting intermediate-range potentials are significantly weaker than the 2π exchange terms, the appearance of new structures might lead to large effects in certain nucleon-deuteron scattering observables. It would be interesting in the future to explore this possibility in a systematic way. Clearly, the N^{4}LO corrections to the short-range part of the 3NF should also be worked out. This work is in progress. Notice that subleading contributions to the three-nucleon contact interactions at N^{4}LO are discussed in Ref. [45]. Finally, the large N^{4}LO corrections for the intermediate-range terms raise an obvious question in regard to whether the chiral expansion for these quantities can be truncated at this order. One should especially keep in mind that while the obtained N^{4}LO corrections do include some of the $1/(m_\Delta - m_N)$ contributions through values of the LECs $c_{2,3,4}$ and, in this sense, take into account physics associated with intermediate excitation of a single Δ isobar, double and triple Δ excitations start contributing only at orders N^{5}LO and N^{6}LO, respectively. Phenomenological studies of Ref. [39] indicate that at least double Δ excitations might induce sizable 3NFs. This issue must be investigated in the future. Rather than calculating N^{5}LO and N^{6}LO corrections to the 3NF in the delta-less formulation of chiral EFT, which correspond to the two-loop level, it is more feasible and probably also more efficient to include the Δ isobar as an explicit degree of freedom in the effective Lagrangian, see Refs. [35, 36, 46–48] for some promising steps in this directions. In such a delta-full formulation, the leading types of $1/(m_\Delta - m_N)$, $1/(m_\Delta - m_N)^2$, ... contributions are resumed and the 3NF mechanisms associated with single, double and triple intermediate Δ-excitations are taken into account already at N^{3}LO. Work along these lines is underway.

Acknowledgments

We are grateful to Ulf-G. Meißner for many valuable comments on the manuscript. This work is supported by the EU HadronPhysics3 project "Study of strongly interacting matter", by the European Research Council (ERC-2010-StG 259218 NuclearEFT) and by the DFG (TR 16, "Subnuclear Structure of Matter").

Appendix A: Ring contributions in momentum space

In this appendix we give the lengthy expressions for subleading contributions to ring diagrams in momentum space. We employ here the general parametrization of local three-body-force:

$$V_{\rm ring} = \sum_{i=1}^{22} \mathcal{G}_i \, \mathcal{R}_i(q_1, \, q_3, \, z), \tag{A.1}$$

where $\mathcal{G}_1, \, \ldots \, , \mathcal{G}_{22}$ are the spin-isospin-momentum operators which we defined in Table I. The N^{4}LO contributions to the structure functions $\mathcal{R}_i(q_1, \, q_3, \, z)$ with $z = \hat{q}_1 \cdot \hat{q}_3$ proportional to g_A^4 are given by

$$\begin{aligned}
\mathcal{R}_1^{(5),g_A^4} \; = \; & \frac{g_A^4}{512 F_\pi^6 \left(z^2-1\right)^2} I(4:0,-q_1,q_3;0)(c_2+c_3) \Big(144 M_\pi^4 \left(z^2-1\right)^2 + 8 M_\pi^2 \left(z^2-1\right) \left(16 z^2 \left(q_1^2+q_3^2\right)\right. \\
& \left. -13 \left(q_1^2+q_3^2\right) + 16 q_1 q_3 z^3 - 10 q_1 q_3 z\right) + q_1^4 \left(32 z^4 - 32 z^2 + 9\right) + 4 q_1^3 q_3 z \left(1-4z^2\right)^2 \\
& + 2 q_1^2 q_3^2 \left(80 z^4 - 78 z^2 + 25\right) + 4 q_1 q_3^3 z \left(1-4z^2\right)^2 + q_3^4 \left(32 z^4 - 32 z^2 + 9\right)\Big) \\
& - \frac{g_A^4}{12288 \pi^2 F_\pi^6 q_1 \left(z^2-1\right)^2} L(q_3)(c_2+c_3) \left(4 M_\pi^2 \left(z^2-1\right) \left(q_1 \left(128 z^2 - 101\right) + 27 q_3 z\right)\right. \\
& \left. + 3 q_1^3 \left(32 z^4 - 32 z^2 + 9\right) + 3 q_1^2 q_3 z \left(32 z^4 - 5\right) + q_1 q_3^2 \left(320 z^4 - 412 z^2 + 173\right) + 3 q_3^3 z \left(26 z^2 - 17\right)\right) \\
& - \frac{g_A^4}{12288 \pi^2 F_\pi^6 q_3 \left(z^2-1\right)^2} L(q_1)(c_2+c_3) \left(4 M_\pi^2 \left(z^2-1\right) \left(27 q_1 z + q_3 \left(128 z^2 - 101\right)\right)\right.
\end{aligned}$$

$$+q_1^3\left(78z^3-51z\right)+q_1^2q_3\left(320z^4-412z^2+173\right)+3q_1q_3^2z\left(32z^4-5\right)+3q_3^3\left(32z^4-32z^2+9\right)\Big)$$
$$-\frac{g_A^4}{12288\pi^2F_\pi^6q_1q_3\left(z^2-1\right)^2}L(q_2)(c_2+c_3)\left(-4M_\pi^2\left(z^2-1\right)\left(27z\left(q_1^2+q_3^2\right)-74q_1q_3z^2+128q_1q_3\right)\right.$$
$$+q_1^4\left(51z-78z^3\right)+4q_1^3q_3\left(2z^4-85z^2+56\right)+2q_1^2q_3^2z\left(116z^4-424z^2+227\right)$$
$$\left.+4q_1q_3^3\left(2z^4-85z^2+56\right)+3q_3^4z\left(17-26z^2\right)\right)-\frac{3g_A^4(c_2+c_3)\left(q_1^2+2q_1q_3z+q_3^2\right)}{4096\pi^2F_\pi^6\left(z^2-1\right)},$$

$$\mathcal{R}_2^{(5),g_A^4} = \frac{g_A^4}{256F_\pi^6\left(z^2-1\right)^2}I(4:0,-q_1,q_3;0)\left(32c_1\left(z^2-1\right)\left(4\left(3M_\pi^2+q_1^2+q_3^2\right)z^2+2q_1q_3z\right.\right.$$
$$\left.-3\left(4M_\pi^2+q_1^2+q_3^2\right)\right)M_\pi^2+c_3\Big(-240\left(z^2-1\right)^2M_\pi^4-24\left(z^2-1\right)\left(8q_1q_3z^3+8\left(q_1^2+q_3^2\right)z^2\right.$$
$$\left.-6q_1q_3z-7\left(q_1^2+q_3^2\right)\right)M_\pi^2+q_1^4\left(-32z^4+48z^2-19\right)+q_3^4\left(-32z^4+48z^2-19\right)$$
$$-4q_1q_3^3z\left(16z^4-20z^2+7\right)-4q_1^3q_3z\left(16z^4-20z^2+7\right)-2q_1^2q_3^2\left(64z^4-90z^2+35\right)\Big)\Big)$$
$$+\frac{g_A^4}{6144F_\pi^6\pi^2\left(4M_\pi^2+q_1^2\right)q_3\left(z^2-1\right)^2\left(-4\left(z^2-1\right)M_\pi^2+q_1^2+q_3^2+2q_1q_3z\right)}L(q_1)$$
$$\times\Big(96c_1\left(z^2-1\right)\left(16\left(z^2-1\right)\left(3q_1z+q_3\left(8z^2-5\right)\right)M_\pi^4+4\left(z\left(5z^2-6\right)q_1^3+q_3\left(12z^4-29z^2+14\right)q_1^2\right.\right.$$
$$\left.+3q_3^2z\left(3-4z^2\right)q_1+q_3^3\left(3-4z^2\right)\right)M_\pi^2-q_1^2\left(zq_1^3+q_3\left(14z^2-11\right)q_1^2+q_3^2z\left(16z^2-13\right)q_1\right.$$
$$\left.\left.+q_3^3\left(4z^2-3\right)\right)\right)M_\pi^2+c_3\Big(-64\left(z^2-1\right)^2\left(45q_1z+q_3\left(256z^2-211\right)\right)M_\pi^6$$
$$-16\left(z^2-1\right)\left(3z\left(53z^2-57\right)q_1^3+q_3\left(848z^4-1673z^2+789\right)q_1^2\right.$$
$$\left.+4q_3^2z\left(48z^4-152z^2+95\right)q_1+4q_3^3\left(48z^4-124z^2+73\right)\right)M_\pi^4-4\left(3z\left(46z^4-110z^2+61\right)q_1^5\right.$$
$$+q_3\left(688z^6-2546z^4+2782z^2-969\right)q_1^4+2q_3^2z\left(144z^6-992z^4+1357z^2-554\right)q_1^3$$
$$+2q_3^3\left(48z^6-640z^4+929z^2-382\right)q_1^2-3q_3^4z\left(96z^4-128z^2+47\right)q_1$$
$$\left.-3q_3^5\left(32z^4-48z^2+19\right)\right)M_\pi^2+q_1^2\left(q_1^2+2q_3zq_1+q_3^2\right)\left(\left(42z^3-33z\right)q_1^3\right.$$
$$\left.+q_3\left(496z^4-884z^2+415\right)q_1^2+3q_3^2z\left(32z^4-32z^2+9\right)q_1+3q_3^3\left(32z^4-48z^2+19\right)\right)\Big)\Big)$$
$$-\frac{g_A^4}{6144F_\pi^6\pi^2q_1q_3\left(4M_\pi^2+q_1^2+q_3^2+2q_1q_3z\right)\left(z^2-1\right)^2\left(-4\left(z^2-1\right)M_\pi^2+q_1^2+q_3^2+2q_1q_3z\right)}L(q_2)$$
$$\times\Big(c_3\Big(-64\left(z^2-1\right)^2\left(-166q_1q_3z^2+45\left(q_1^2+q_3^2\right)z+256q_1q_3\right)M_\pi^6+16\left(z^2-1\right)\left(3z\left(57-53z^2\right)q_1^4\right.$$
$$+4q_3\left(53z^4-281z^2+240\right)q_1^3+2q_3^2z\left(434z^4-1433z^2+1035\right)q_1^2+4q_3^3\left(53z^4-281z^2+240\right)q_1$$
$$\left.+3q_3^4z\left(57-53z^2\right)\right)M_\pi^4-4\left(q_1^2+2q_3zq_1+q_3^2\right)\left(3z\left(46z^4-110z^2+61\right)q_1^4\right.$$
$$-4q_3\left(34z^6-314z^4+577z^2-288\right)q_1^3+2q_3^2z\left(-268z^6+1574z^4-2476z^2+1143\right)q_1^2$$
$$\left.-4q_3^3\left(34z^6-314z^4+577z^2-288\right)q_1+3q_3^4z\left(46z^4-110z^2+61\right)\right)M_\pi^2$$
$$+\left(q_1^2+2q_3zq_1+q_3^2\right)^2\left(\left(42z^3-33z\right)q_1^4-4q_3\left(82z^4-203z^2+112\right)q_1^3\right.$$
$$\left.-2q_3^2z\left(364z^4-824z^2+433\right)q_1^2-4q_3^3\left(82z^4-203z^2+112\right)q_1+3q_3^4z\left(14z^2-11\right)\right)\Big)$$
$$-96c_1M_\pi^2\left(z^2-1\right)\left(-16\left(z^2-1\right)\left(3zq_1^2+2q_3\left(8-5z^2\right)q_1+3q_3^2z\right)M_\pi^4+4\left(z\left(6-5z^2\right)q_1^4\right.\right.$$
$$\left.+4q_3\left(z^4-8z^2+8\right)q_1^3+2q_3^2z\left(10z^4-41z^2+34\right)q_1^2+4q_3^3\left(z^4-8z^2+8\right)q_1+q_3^4z\left(6-5z^2\right)\right)M_\pi^2$$
$$\left.+\left(q_1^2+2q_3zq_1+q_3^2\right)^2\left(zq_1^2+2q_3\left(8-7z^2\right)q_1+q_3^2z\right)\right)\Big)$$
$$+\frac{g_A^4}{6144F_\pi^6\pi^2q_1\left(4M_\pi^2+q_3^2\right)\left(z^2-1\right)^2\left(-4\left(z^2-1\right)M_\pi^2+q_1^2+q_3^2+2q_1q_3z\right)}L(q_3)$$
$$\times\Big(96c_1\left(z^2-1\right)\left(16\left(z^2-1\right)\left(3q_3z+q_1\left(8z^2-5\right)\right)M_\pi^4-4\left(\left(4z^2-3\right)q_1^3+3q_3z\left(4z^2-3\right)q_1^2\right.\right.$$
$$\left.+q_3^2\left(-12z^4+29z^2-14\right)q_1+q_3^3z\left(6-5z^2\right)\right)M_\pi^2-q_3^2\left(\left(4z^2-3\right)q_1^3+q_3z\left(16z^2-13\right)q_1^2\right.$$
$$\left.\left.+q_3^2\left(14z^2-11\right)q_1+q_3^3z\right)\right)M_\pi^2+c_3\Big(-64\left(z^2-1\right)^2\left(45q_3z+q_1\left(256z^2-211\right)\right)M_\pi^6$$
$$-16\left(z^2-1\right)\left(4\left(48z^4-124z^2+73\right)q_1^3+4q_3z\left(48z^4-152z^2+95\right)q_1^2\right.$$

$$
\begin{aligned}
& +q_3^2\left(848z^4-1673z^2+789\right)q_1+3q_3^3z\left(53z^2-57\right)\right)M_\pi^4+4\left(3\left(32z^4-48z^2+19\right)q_1^5\right.\\
& +3q_3z\left(96z^4-128z^2+47\right)q_1^4+2q_3^2\left(-48z^6+640z^4-929z^2+382\right)q_1^3\\
& +2q_3^3z\left(-144z^6+992z^4-1357z^2+554\right)q_1^2+q_3^4\left(-688z^6+2546z^4-2782z^2+969\right)q_1\\
& -3q_3^5z\left(46z^4-110z^2+61\right)\right)M_\pi^2+q_3^2\left(q_1^2+2q_3zq_1+q_3^2\right)\left(3\left(32z^4-48z^2+19\right)q_1^3\right.\\
& \left.\left.+3q_3z\left(32z^4-32z^2+9\right)q_1^2+q_3^2\left(496z^4-884z^2+415\right)q_1+3q_3^3z\left(14z^2-11\right)\right)\right)\\
& +\frac{c_3\left(q_1^2+2q_3zq_1+q_3^2\right)g_A^4}{2048F_\pi^6\pi^2\left(z^2-1\right)},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_3^{(5),g_A^4} &= -\frac{3g_A^4}{16F_\pi^6\left(z^2-1\right)}I(4:0,-q_1,q_3;0)c_3q_1q_3(q_1z+q_3)(q_1+q_3z)\\
&+\frac{3g_A^4}{128\pi^2F_\pi^6\left(z^2-1\right)\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}q_1L(q_1)\left(8c_1M_\pi^2\left(z^2-1\right)(q_1+q_3z)\right.\\
&\left.+c_3\left(-8M_\pi^2\left(z^2-1\right)\left(q_1+q_3z^3\right)-\left(q_1^2+2q_1q_3z+q_3^2\right)\left(2q_1z^2-3q_1-q_3z\right)\right)\right)\\
&+\frac{3g_A^4}{128\pi^2F_\pi^6\left(z^2-1\right)\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}q_3L(q_3)\left(8c_1M_\pi^2\left(z^2-1\right)(q_1z+q_3)\right.\\
&\left.+c_3\left(\left(q_1^2+2q_1q_3z+q_3^2\right)\left(q_1z-2q_3z^2+3q_3\right)-8M_\pi^2\left(z^2-1\right)\left(q_1z^3+q_3\right)\right)\right)\\
&-\frac{3g_A^4}{128\pi^2F_\pi^6\left(z^2-1\right)\left(4M_\pi^2+q_1^2+2q_1q_3z+q_3^2\right)\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}L(q_2)\\
&\times\left(8c_1M_\pi^2\left(z^2-1\right)\left(4M_\pi^2\left(q_1\left(q_1-q_3z^3+3q_3z\right)+q_3^2\right)+\left(q_1^2+2q_1q_3z+q_3^2\right)^2\right)\right.\\
&+c_3\left(-32M_\pi^4\left(z^2-1\right)\left(q_1^2-q_1q_3z\left(z^2-3\right)+q_3^2\right)-4M_\pi^2\left(q_1^2+2q_1q_3z+q_3^2\right)\left(q_1^2\left(4z^2-5\right)\right.\right.\\
&\left.+q_1q_3z\left(-3z^4+15z^2-14\right)+q_3^2\left(4z^2-5\right)\right)-\left(q_1^2+2q_1q_3z+q_3^2\right)^2\left(2z^2\left(q_1^2+q_3^2\right)-3\left(q_1^2+q_3^2\right)\right.\\
&\left.\left.\left.+5q_1q_3z^3-7q_1q_3z\right)\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_4^{(5),g_A^4} &= \frac{g_A^4}{8F_\pi^6\left(z^2-1\right)}I(4:0,-q_1,q_3;0)q_1q_3(c_2+c_3)(q_1z+q_3)(q_1+q_3z)\\
&+\frac{g_A^4}{64\pi^2F_\pi^6\left(z^2-1\right)}L(q_2)(c_2+c_3)\left(q_1^2-q_1q_3z\left(z^2-3\right)+q_3^2\right)-\frac{g_A^4q_1L(q_1)(c_2+c_3)(q_1+q_3z)}{64\pi^2F_\pi^6\left(z^2-1\right)}\\
&-\frac{g_A^4q_3L(q_3)(c_2+c_3)(q_1z+q_3)}{64\pi^2F_\pi^6\left(z^2-1\right)},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_5^{(5),g_A^4} &= -\frac{g_A^4}{8F_\pi^6\left(z^2-1\right)}I(4:0,-q_1,q_3;0)c_4q_1z(q_1z+q_3)\left(q_1^2+2q_1q_3z+q_3^2\right)\\
&+\frac{g_A^4}{64\pi^2F_\pi^6q_3\left(z^2-1\right)\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}c_4q_1L(q_1)\left(\left(q_1^2+2q_1q_3z+q_3^2\right)\left(z\left(q_1^2+q_3^2\right)\right.\right.\\
&\left.\left.+4q_1q_3z^2-2q_1q_3\right)-8M_\pi^2\left(z^2-1\right)\left(q_1^2z+q_1q_3\left(3z^2-1\right)+q_3^2z^3\right)\right)\\
&+\frac{g_A^4}{64\pi^2F_\pi^6q_3\left(z^2-1\right)\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}c_4L(q_2)\left(8M_\pi^2\left(z^2-1\right)\left(q_1^3z+q_1^2q_3\left(z^2+2\right)\right.\right.\\
&\left.\left.-q_1q_3^2z\left(z^2-4\right)+q_3^3\right)-\left(q_1^2+2q_1q_3z+q_3^2\right)^2\left(q_1z+q_3\left(3-2z^2\right)\right)\right)\\
&+\frac{g_A^4}{64\pi^2F_\pi^6\left(z^2-1\right)\left(4M_\pi^2+q_3^2\right)\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}c_4L(q_3)\left(-32M_\pi^4\left(z^2-1\right)\right.\\
&\times\left(q_1^2z^2+q_1q_3z\left(z^2+1\right)+q_3^2\right)+4M_\pi^2\left(q_1^4z^2+q_1^3q_3\left(3z^3+z\right)+q_1^2q_3^2\left(-z^4+5z^2+2\right)\right.\\
&\left.-q_1q_3^3z\left(3z^4+z^2-8\right)+q_3^4\left(5-4z^2\right)\right)+q_3^2\left(q_1^2+2q_1q_3z+q_3^2\right)\left(q_1^2z^2+q_1q_3z\left(z^2+1\right)\right.\\
&\left.\left.+q_3^2\left(3-2z^2\right)\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_7^{(5),g_A^4} &= \frac{g_A^4}{128F_\pi^6q_1q_3\left(z^2-1\right)^2}I(4:0,-q_1,q_3;0)c_4z\left(q_1^2+2q_3zq_1+q_3^2\right)\left(4\left(3M_\pi^2+q_1^2+q_3^2\right)z^2+2q_1q_3z\right.\\
&\left.-3\left(4M_\pi^2+q_1^2+q_3^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{g_A^4}{3072F_\pi^6\pi^2 q_1\left(4M_\pi^2+q_1^2\right)q_3^2\left(z^2-1\right)^2\left(-4\left(z^2-1\right)M_\pi^2+q_1^2+q_3^2+2q_1q_3z\right)}c_4L(q_1)\\
&\times\Big(-1024(q_1+q_3z)\left(z^2-1\right)^2M_\pi^6-16\left(z^2-1\right)\left(\left(41z^2-48\right)q_1^3+q_3z\left(50z^2-71\right)q_1^2\right.\\
&+q_3^2\left(24z^4-29z^2-16\right)q_1+q_3^3z\left(24z^2-31\right)\big)M_\pi^4-4\left(\left(25z^4-70z^2+42\right)q_1^5\right.\\
&+q_3z\left(40z^4-143z^2+88\right)q_1^4+q_3^2\left(36z^6-115z^4+17z^2+32\right)q_1^3+q_3^3z\left(12z^4-101z^2+59\right)q_1^2\\
&+3q_3^4z^2\left(7-12z^2\right)q_1+3q_3^5z\left(3-4z^2\right)\big)M_\pi^2+q_1^2\left(q_1^2+2q_3zq_1+q_3^2\right)\left(\left(13z^2-10\right)q_1^3\right.\\
&+q_3z\left(16z^2-7\right)q_1^2+3q_3^2z^2\left(4z^2-1\right)q_1+3q_3^3z\left(4z^2-3\right)\big)\Big)\\
&-\frac{g_A^4}{3072F_\pi^6\pi^2q_1^2q_3^2\left(z^2-1\right)^2\left(-4\left(z^2-1\right)M_\pi^2+q_1^2+q_3^2+2q_1q_3z\right)}c_4L(q_2)\left(256\left(q_1^2+2q_3zq_1+q_3^2\right)\right.\\
&\times\left(z^2-1\right)^2M_\pi^4+4\left(z^2-1\right)\left(\left(25z^2-32\right)q_1^4+4q_3z\left(13z^2-20\right)q_1^3+2q_3^2\left(14z^4-15z^2-20\right)q_1^2\right.\\
&+4q_3^3z\left(13z^2-20\right)q_1+q_3^4\left(25z^2-32\right)\big)M_\pi^2-\left(q_1^2+2q_3zq_1+q_3^2\right)^2\left(2q_1q_3z^3+13\left(q_1^2+q_3^2\right)z^2\right.\\
&+4q_1q_3z-10\left(q_1^2+q_3^2\right)\big)\big)\\
&-\frac{g_A^4}{3072F_\pi^6\pi^2q_1^2q_3\left(4M_\pi^2+q_3^2\right)\left(z^2-1\right)^2\left(-4\left(z^2-1\right)M_\pi^2+q_1^2+q_3^2+2q_1q_3z\right)}c_4L(q_3)\\
&\times\Big(-1024(q_3+q_1z)\left(z^2-1\right)^2M_\pi^6-16\left(z^2-1\right)\left(z\left(24z^2-31\right)q_1^3+q_3\left(24z^4-29z^2-16\right)q_1^2\right.\\
&+q_3^2z\left(50z^2-71\right)q_1+q_3^3\left(41z^2-48\right)\big)M_\pi^4+4\left(3z\left(4z^2-3\right)q_1^5+3q_3z^2\left(12z^2-7\right)q_1^4\right.\\
&+q_3^2z\left(-12z^4+101z^2-59\right)q_1^3-q_3^3\left(36z^6-115z^4+17z^2+32\right)q_1^2+q_3^4z\left(-40z^4+143z^2-88\right)q_1\\
&+q_3^5\left(-25z^4+70z^2-42\right)\big)M_\pi^2+q_3^2\left(q_1^2+2q_3zq_1+q_3^2\right)\left(3z\left(4z^2-3\right)q_1^3+3q_3z^2\left(4z^2-1\right)q_1^2\right.\\
&+q_3^2z\left(16z^2-7\right)q_1+q_3^3\left(13z^2-10\right)\big)\Big)-\frac{c_4\left(2q_1q_3+\left(q_1^2+q_3^2\right)z\right)g_A^4}{3072F_\pi^6\pi^2q_1q_3\left(z^2-1\right)},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_8^{(5),g_A^4} \;=\; &\frac{3g_A^4}{16F_\pi^6q_1\left(z^2-1\right)^2}I(4:0,-q_1,q_3;0)q_3\left(c_3\left(4z^3\left(3M_\pi^2+q_1^2+q_3^2\right)-12M_\pi^2z-7z\left(q_1^2+q_3^2\right)+8q_1q_3z^4\right.\right.\\
&\left.-12q_1q_3z^2-2q_1q_3\right)-16c_1M_\pi^2z\left(z^2-1\right)\big)\\
&-\frac{3g_A^4}{128\pi^2F_\pi^6q_1\left(z^2-1\right)^2\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}L(q_1)\left(c_3\left(4M_\pi^2\left(z^2-1\right)\left(q_1\left(z^2+4\right)\right.\right.\right.\\
&\left.+5q_3z\right)+\left(q_1^2+2q_1q_3z+q_3^2\right)\left(3q_1\left(z^2-2\right)+q_3z\left(4z^2-7\right)\right)\big)-16c_1M_\pi^2\left(z^2-1\right)\left(q_1+q_3z\right)\big)\\
&-\frac{3g_A^4}{128\pi^2F_\pi^6q_1^2\left(z^2-1\right)^2\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}q_3L(q_3)\left(c_3\left(4M_\pi^2\left(z^2-1\right)\left(5q_1z\right.\right.\right.\\
&\left.+q_3\left(z^2+4\right)\right)+\left(q_1^2+2q_1q_3z+q_3^2\right)\left(q_1z\left(4z^2-7\right)+3q_3\left(z^2-2\right)\right)\big)-16c_1M_\pi^2\left(z^2-1\right)\left(q_1z+q_3\right)\big)\\
&-\frac{3g_A^4}{128\pi^2F_\pi^6q_1^2\left(z^2-1\right)^2\left(4M_\pi^2+q_1^2+2q_1q_3z+q_3^2\right)\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}L(q_2)\\
&\times\Big(16c_1M_\pi^2\left(z^2-1\right)\Big(4M_\pi^2\left(q_1\left(q_1-q_3z^3+3q_3z\right)+q_3^2\right)+\left(q_1^2+2q_1q_3z+q_3^2\right)^2\Big)\\
&+c_3\left(-16M_\pi^4\left(z^2-1\right)\left(q_1^2\left(z^2+4\right)-2q_1q_3z\left(z^2-6\right)+q_3^2\left(z^2+4\right)\right)-4M_\pi^2\left(q_1^2+2q_1q_3z+q_3^2\right)\right.\\
&\times\left(q_1^2\left(z^4+6z^2-10\right)-2q_1q_3z\left(2z^4-13z^2+14\right)+q_3^2\left(z^4+6z^2-10\right)\right)\\
&\left.-\left(q_1^2+2q_1q_3z+q_3^2\right)^2\left(3q_1^2\left(z^2-2\right)+2q_1q_3z\left(4z^2-7\right)+3q_3^2\left(z^2-2\right)\right)\right)\Big)+\frac{3g_A^4c_3q_3z}{128\pi^2F_\pi^6q_1\left(z^2-1\right)},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_9^{(5),g_A^4} \;=\; &\frac{3g_A^4}{32F_\pi^6\left(z^2-1\right)^2}I(4:0,-q_1,q_3;0)\left(16c_1M_\pi^2\left(z^2-1\right)+c_3\left(-12M_\pi^2\left(z^2-1\right)-2z^2\left(q_1^2+3q_1q_3z+q_3^2\right)\right.\right.\\
&\left.+5\left(q_1^2+q_3^2\right)+12q_1q_3z\right)\big)\\
&-\frac{3g_A^4}{256\pi^2F_\pi^6q_3\left(z^2-1\right)^2\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}L(q_1)\left(16c_1M_\pi^2z\left(z^2-1\right)\left(q_1+q_3z\right)\right.\\
&+c_3\left(-4M_\pi^2\left(z^2-1\right)\left(5q_1z+q_3\left(4z^4-4z^2+5\right)\right)-\left(q_1^2+2q_1q_3z+q_3^2\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(4q_1z^3-7q_1z+2q_3z^2-5q_3\right)\big)\big) \\
&-\frac{3g_A^4}{256\pi^2F_\pi^6q_1\left(z^2-1\right)^2\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}L(q_3)\left(16c_1M_\pi^2z\left(z^2-1\right)(q_1z+q_3)\right. \\
&+c_3\left(-4M_\pi^2\left(z^2-1\right)\left(q_1\left(4z^4-4z^2+5\right)+5q_3z\right)-\left(q_1^2+2q_1q_3z+q_3^2\right)\right. \\
&\times\left(2q_1z^2-5q_1+4q_3z^3-7q_3z\right)\big)\big) \\
&+\frac{3g_A^4}{256\pi^2F_\pi^6q_1q_3\left(z^2-1\right)^2\left(4M_\pi^2+q_1^2+2q_1q_3z+q_3^2\right)\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}zL(q_2) \\
&\times\Big(16c_1M_\pi^2\left(z^2-1\right)\left(4M_\pi^2\left(q_1\left(q_1-q_3z^3+3q_3z\right)+q_3^2\right)+\left(q_1^2+2q_1q_3z+q_3^2\right)^2\right) \\
&+c_3\left(-16M_\pi^4\left(z^2-1\right)\left(5\left(q_1^2+q_3^2\right)-4q_1q_3z^3+14q_1q_3z\right)-4M_\pi^2\left(3z^2-4\right)\left(q_1^2+2q_1q_3z+q_3^2\right)\right. \\
&\times\left(3\left(q_1^2+q_3^2\right)-2q_1q_3z^3+8q_1q_3z\right)-\left(q_1^2+2q_1q_3z+q_3^2\right)^2\left(4z^2\left(q_1^2+q_3^2\right)-7\left(q_1^2+q_3^2\right)\right. \\
&+10q_1q_3z^3-16q_1q_3z\big)\big)\Big)-\frac{3g_A^4c_3}{256\pi^2F_\pi^6\left(z^2-1\right)},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_{10}^{(5),g_A^4} &= \frac{3g_A^4}{32F_\pi^6\left(z^2-1\right)^2}I(4:0,-q_1,q_3;0)z\left(16c_1M_\pi^2z\left(z^2-1\right)+c_3\left(-4z^3\left(3M_\pi^2+q_1^2+q_3^2\right)+12M_\pi^2z\right.\right. \\
&+7z\left(q_1^2+q_3^2\right)-8q_1q_3z^4+12q_1q_3z^2+2q_1q_3\big)\big) \\
&-\frac{3g_A^4}{256\pi^2F_\pi^6q_3\left(z^2-1\right)^2\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}zL(q_1)\left(16c_1M_\pi^2\left(z^2-1\right)(q_1+q_3z)\right. \\
&+c_3\left(-4M_\pi^2\left(z^2-1\right)\left(q_1\left(z^2+4\right)+5q_3z\right)-\left(q_1^2+2q_1q_3z+q_3^2\right)\left(3q_1\left(z^2-2\right)+q_3z\left(4z^2-7\right)\right)\right)\big) \\
&-\frac{3g_A^4}{256\pi^2F_\pi^6q_1\left(z^2-1\right)^2\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}zL(q_3)\left(16c_1M_\pi^2\left(z^2-1\right)(q_1z+q_3)\right. \\
&+c_3\left(-4M_\pi^2\left(z^2-1\right)\left(5q_1z+q_3\left(z^2+4\right)\right)-\left(q_1^2+2q_1q_3z+q_3^2\right)\left(q_1z\left(4z^2-7\right)+3q_3\left(z^2-2\right)\right)\right)\big) \\
&+\frac{3g_A^4}{256\pi^2F_\pi^6q_1q_3\left(z^2-1\right)^2\left(4M_\pi^2+q_1^2+2q_1q_3z+q_3^2\right)\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}zL(q_2) \\
&\times\Big(16c_1M_\pi^2\left(z^2-1\right)\left(4M_\pi^2\left(q_1\left(q_1-q_3z^3+3q_3z\right)+q_3^2\right)+\left(q_1^2+2q_1q_3z+q_3^2\right)^2\right) \\
&+c_3\left(-16M_\pi^4\left(z^2-1\right)\left(q_1^2\left(z^2+4\right)-2q_1q_3z\left(z^2-6\right)+q_3^2\left(z^2+4\right)\right)-4M_\pi^2\left(q_1^2+2q_1q_3z+q_3^2\right)\right. \\
&\times\left(q_1^2\left(z^4+6z^2-10\right)-2q_1q_3z\left(2z^4-13z^2+14\right)+q_3^2\left(z^4+6z^2-10\right)\right) \\
&-\left(q_1^2+2q_1q_3z+q_3^2\right)^2\left(3q_1^2\left(z^2-2\right)+2q_1q_3z\left(4z^2-7\right)+3q_3^2\left(z^2-2\right)\right)\big)\Big)-\frac{3g_A^4c_3}{256\pi^2F_\pi^6\left(z^2-1\right)},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_{11}^{(5),g_A^4} &= -\frac{g_A^4}{16F_\pi^6q_1\left(z^2-1\right)^2}I(4:0,-q_1,q_3;0)c_4z(q_1z+q_3)\left(-12M_\pi^2\left(z^2-1\right)+3q_1^2+6q_1q_3z+q_3^2\left(7-4z^2\right)\right) \\
&-\frac{g_A^4}{128\pi^2F_\pi^6q_1q_3\left(z^2-1\right)^2\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}c_4L(q_1)\left(4M_\pi^2\left(z^2-1\right)\left(q_1^2z\left(z^2+4\right)\right.\right. \\
&-2q_1q_3\left(2z^4-9z^2+2\right)+5q_3^2z\big)-\left(q_1^2+2q_1q_3z+q_3^2\right)\left(q_1^2z\left(z^2+2\right)-2q_1q_3\left(2z^4-7z^2+2\right)\right. \\
&+q_3^2z\left(7-4z^2\right)\big)\big) \\
&-\frac{g_A^4}{128\pi^2F_\pi^6q_1^2q_3\left(z^2-1\right)^2\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}c_4L(q_2)\left(\left(q_1^2+2q_1q_3z+q_3^2\right)\right. \\
&\times\left(q_1^3z\left(z^2+2\right)+3q_1^2q_3\left(2z^4-z^2+2\right)-q_1q_3^2z\left(z^2-10\right)-3q_3^3\left(z^2-2\right)\right) \\
&-4M_\pi^2\left(z^2-1\right)\left(q_1^3z\left(z^2+4\right)+q_1^2q_3\left(6z^4+z^2+8\right)+3q_1q_3^2z\left(z^2+4\right)+q_3^3\left(z^2+4\right)\right)\big) \\
&-\frac{g_A^4}{128\pi^2F_\pi^6q_1^2\left(z^2-1\right)^2\left(4M_\pi^2+q_3^2\right)\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)}c_4L(q_3)\left(16M_\pi^4\left(z^2-1\right)\right. \\
&\times\left(5q_1^2z^2+5q_1q_3z\left(z^2+1\right)+q_3^2\left(z^2+4\right)\right)-4M_\pi^2\left(3q_1^4z^2+3q_1^3q_3\left(3z^3+z\right)+q_1^2q_3^2\left(-z^4+15z^2+4\right)\right. \\
&+q_1q_3^3z\left(-7z^4+z^2+18\right)-q_3^4\left(z^4+6z^2-10\right)\big)-3q_3^2\left(q_1^2+2q_1q_3z+q_3^2\right)\left(q_1^2z^2+q_1q_3z\left(z^2+1\right)\right.
\end{aligned}
$$

$$-q_3^2\left(z^2-2\right)\right)\right)+\frac{g_A^4 c_4(q_1+q_3 z)}{128\pi^2 F_\pi^6 q_1\left(z^2-1\right)},$$

$$\begin{aligned}
\mathcal{R}_{12}^{(5),g_A^4} &= -\frac{g_A^4}{16F_\pi^6 q_3\left(z^2-1\right)^2} I(4:0,-q_1,q_3;0) c_4 z\left(12M_\pi^2\left(z^2-1\right)(q_1+q_3 z)+q_1^3\left(-\left(2z^2+1\right)\right)\right.\\
&\left.-q_1^2 q_3 z\left(4z^2+5\right)-q_1 q_3^2\left(4z^2+5\right)+q_3^3 z\left(4z^2-7\right)\right)\\
&-\frac{g_A^4}{128\pi^2 F_\pi^6 q_3^2\left(z^2-1\right)^2\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)} c_4 z L(q_1)\left(\left(q_1^2+2q_1q_3z+q_3^2\right)\right.\\
&\times\left(3q_1^2 z+q_1\left(5q_3z^2+q_3\right)+q_3^2 z\left(7-4z^2\right)\right)-4M_\pi^2\left(z^2-1\right)\left(5z\left(q_1^2+q_3^2\right)+9q_1q_3z^2+q_1q_3\right)\Big)\\
&-\frac{g_A^4}{128\pi^2 F_\pi^6 q_1 q_3^2\left(z^2-1\right)^2\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)} c_4 z L(q_2)\left(q_1^2+2q_1q_3z+q_3^2\right)\\
&\times\left(4M_\pi^2\left(z^2-1\right)\left(5q_1z+q_3\left(z^2+4\right)\right)-3q_1^3 z-q_1^2 q_3\left(7z^2+2\right)+q_1q_3^2 z\left(2z^2-11\right)+3q_3^3\left(z^2-2\right)\right)\\
&-\frac{g_A^4}{128\pi^2 F_\pi^6 q_1 q_3\left(z^2-1\right)^2\left(4M_\pi^2+q_3^2\right)\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)} c_4 z L(q_3)\\
&\times\Big(-16M_\pi^4\left(z^2-1\right)\left(q_1^2\left(4z^2+1\right)+2q_1q_3z\left(2z^2+3\right)+q_3^2\left(z^2+4\right)\right)+4M_\pi^2\left(q_1^4\left(2z^2+1\right)\right.\\
&\left.+6q_1^3q_3z\left(z^2+1\right)-2q_1^2q_3^2\left(z^4-7z^2-3\right)-2q_1q_3^3z\left(3z^4+z^2-10\right)-q_3^4\left(z^4+6z^2-10\right)\right)\\
&+q_3^2\left(q_1^2+2q_1q_3z+q_3^2\right)\left(q_1^2\left(2z^2+1\right)+2q_1q_3z\left(z^2+2\right)-3q_3^2\left(z^2-2\right)\right)\Big)-\frac{g_A^4 c_4(q_1 z+q_3)}{128\pi^2 F_\pi^6 q_3\left(z^2-1\right)},\\
\mathcal{R}_{13}^{(5),g_A^4} &= -\frac{g_A^4}{16F_\pi^6 q_3\left(z^2-1\right)^2} I(4:0,-q_1,q_3;0) c_4(q_1z+q_3)\left(12M_\pi^2\left(z^2-1\right)-q_1^2\left(2z^2+1\right)\right.\\
&\left.-2q_1q_3\left(2z^3+z\right)+q_3^2\left(2z^2-5\right)\right)\\
&-\frac{g_A^4}{128\pi^2 F_\pi^6 q_3^2\left(z^2-1\right)^2\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)} c_4 L(q_1)\left(\left(q_1^2+2q_1q_3z+q_3^2\right)\right.\\
&\times\left(3q_1^2z^2+2q_1q_3\left(2z^3+z\right)+q_3^2\left(5-2z^2\right)\right)-4M_\pi^2\left(z^2-1\right)\left(5q_1^2z^2+2q_1q_3\left(4z^3+z\right)\right.\\
&\left.\left.+q_3^2\left(4z^4-4z^2+5\right)\right)\right)\\
&-\frac{g_A^4}{128\pi^2 F_\pi^6 q_1 q_3^2\left(z^2-1\right)^2\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)} c_4 L(q_2)\left(4M_\pi^2\left(z^2-1\right)\left(5q_1^3z^2\right.\right.\\
&\left.+5q_1^2q_3\left(2z^3+z\right)+q_1q_3^2\left(-4z^4+23z^2-4\right)+5q_3^3z\right)-\left(q_1^2+2q_1q_3z+q_3^2\right)\left(3q_1^3z^2+3q_1^2q_3\left(2z^3+z\right)\right.\\
&\left.\left.+q_1q_3^2\left(-8z^4+21z^2-4\right)+q_3^3z\left(7-4z^2\right)\right)\right)\\
&-\frac{g_A^4}{128\pi^2 F_\pi^6 q_1 q_3\left(z^2-1\right)^2\left(4M_\pi^2+q_3^2\right)\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)} c_4 L(q_3)\left(-16M_\pi^4\left(z^2-1\right)\right.\\
&\times\left(q_1^2\left(4z^3+z\right)+q_1q_3\left(4z^4+5z^2+1\right)+5q_3^2z\right)+4M_\pi^2\left(q_1^4\left(2z^3+z\right)+q_1^3q_3\left(6z^4+5z^2+1\right)\right.\\
&\left.+q_1^2q_3^2z\left(-2z^4+11z^2+9\right)+q_1q_3^3\left(-6z^6-3z^4+19z^2+2\right)+3q_3^4z\left(4-3z^2\right)\right)\\
&\left.+q_3^2\left(q_1^2+2q_1q_3z+q_3^2\right)\left(q_1^2\left(2z^3+z\right)+q_1q_3\left(2z^4+3z^2+1\right)+q_3^2z\left(7-4z^2\right)\right)\right)\\
&-\frac{g_A^4 c_4(q_1z+q_3)}{128\pi^2 F_\pi^6 q_3\left(z^2-1\right)},\\
\mathcal{R}_{14}^{(5),g_A^4} &= -\frac{g_A^4}{16F_\pi^6 q_3^2\left(z^2-1\right)^2} I(4:0,-q_1,q_3;0) c_4 q_1\left(-12M_\pi^2\left(z^2-1\right)(q_1+q_3z)+q_1^3\left(2z^2+1\right)\right.\\
&\left.+q_1^2q_3z\left(4z^2+5\right)+q_1q_3^2\left(4z^2+5\right)+q_3^3z\left(7-4z^2\right)\right)\\
&+\frac{g_A^4}{128\pi^2 F_\pi^6 q_3^3\left(z^2-1\right)^2\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)} c_4 q_1 L(q_1)\left(\left(q_1^2+2q_1q_3z+q_3^2\right)\right.\\
&\times\left(3q_1^2z+q_1\left(5q_3z^2+q_3\right)+q_3^2z\left(7-4z^2\right)\right)-4M_\pi^2\left(z^2-1\right)\left(5z\left(q_1^2+q_3^2\right)+9q_1q_3z^2+q_1q_3\right)\Big)\\
&-\frac{g_A^4}{128\pi^2 F_\pi^6 q_3^3\left(z^2-1\right)^2\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)} c_4 L(q_2)\left(q_1^2+2q_1q_3z+q_3^2\right)\\
&\times\left(-4M_\pi^2\left(z^2-1\right)\left(5q_1z+q_3\left(z^2+4\right)\right)+3q_1^3z+q_1^2q_3\left(7z^2+2\right)+q_1q_3^2z\left(11-2z^2\right)-3q_3^3\left(z^2-2\right)\right)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{g_A^4}{128\pi^2 F_\pi^6 q_3^2 \left(z^2-1\right)^2 \left(4M_\pi^2+q_3^2\right)\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)} c_4 L(q_3)\left(16M_\pi^4\left(z^2-1\right)\right.\\
& \times\left(q_1^2\left(4z^2+1\right)+2q_1q_3z\left(2z^2+3\right)+q_3^2\left(z^2+4\right)\right)-4M_\pi^2\left(q_1^4\left(2z^2+1\right)+6q_1^3q_3z\left(z^2+1\right)\right.\\
& \left.-2q_1^2q_3^2\left(z^4-7z^2-3\right)-2q_1q_3^3z\left(3z^4+z^2-10\right)-q_3^4\left(z^4+6z^2-10\right)\right)-q_3^2\left(q_1^2+2q_1q_3z+q_3^2\right)\\
& \left.\times\left(q_1^2\left(2z^2+1\right)+2q_1q_3z\left(z^2+2\right)-3q_3^2\left(z^2-2\right)\right)\right)+\frac{g_A^4 c_4 q_1\left(q_1+q_3z\right)}{128\pi^2F_\pi^6q_3^2\left(z^2-1\right)},\\
\mathcal{R}_{15}^{(5),g_A^4} \;=\; & -\frac{g_A^4}{16F_\pi^6\left(z^2-1\right)^2} I(4:0,-q_1,q_3;0)(c_2+c_3)\left(2z^2\left(-2M_\pi^2+q_1^2+q_3^2\right)+4M_\pi^2+q_1^2\right.\\
& \left.+2q_1q_3z^3+4q_1q_3z+q_3^2\right)\\
& -\frac{g_A^4 zL(q_2)(c_2+c_3)\left(3q_1^2-2q_1q_3z\left(z^2-4\right)+3q_3^2\right)}{128\pi^2F_\pi^6q_1q_3\left(z^2-1\right)^2}+\frac{g_A^4L(q_1)(c_2+c_3)\left(3q_1z+2q_3z^2+q_3\right)}{128\pi^2F_\pi^6q_3\left(z^2-1\right)^2}\\
& +\frac{g_A^4L(q_3)(c_2+c_3)\left(2q_1z^2+q_1+3q_3z\right)}{128\pi^2F_\pi^6q_1\left(z^2-1\right)^2}+\frac{g_A^4(c_2+c_3)}{128\pi^2F_\pi^6\left(z^2-1\right)},\\
\mathcal{R}_{16}^{(5),g_A^4} \;=\; & -\frac{g_A^4}{8F_\pi^6q_1q_3\left(z^2-1\right)^2} I(4:0,-q_1,q_3;0)z(c_2+c_3)\left(q_1^2+2q_1q_3z+q_3^2\right)\\
& \times\left(-4M_\pi^2\left(z^2-1\right)+3q_1(q_1+2q_3z)+2q_3^2z^2+q_3^2\right)\\
& -\frac{g_A^4L(q_2)(c_2+c_3)\left(q_1^2+2q_1q_3z+q_3^2\right)\left(q_1^2\left(z^2+2\right)+2q_1q_3z\left(z^2+2\right)+3q_3^2z^2\right)}{64\pi^2F_\pi^6q_1^2q_3^2\left(z^2-1\right)^2}\\
& +\frac{g_A^4L(q_1)(c_2+c_3)\left(q_1^3\left(z^2+2\right)+q_1^2q_3z\left(4z^2+5\right)+q_1q_3^2z^2\left(2z^2+7\right)+q_3^3z\left(2z^2+1\right)\right)}{64\pi^2F_\pi^6q_1q_3^2\left(z^2-1\right)^2}\\
& +\frac{3g_A^4zL(q_3)(c_2+c_3)(q_1+q_3z)\left(q_1^2+2q_1q_3z+q_3^2\right)}{64\pi^2F_\pi^6q_1^2q_3\left(z^2-1\right)^2}+\frac{g_A^4(c_2+c_3)\left(z\left(q_1^2+q_3^2\right)+2q_1q_3\right)}{64\pi^2F_\pi^6q_1q_3\left(z^2-1\right)},\\
\mathcal{R}_{17}^{(5),g_A^4} \;=\; & -\frac{g_A^4}{16F_\pi^6\left(z^2-1\right)} I(4:0,-q_1,q_3;0)(c_2+c_3)\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right)\\
& -\frac{g_A^4 zL(q_2)(c_2+c_3)\left(q_1^2+2q_1q_3z+q_3^2\right)}{128\pi^2F_\pi^6q_1q_3\left(z^2-1\right)}+\frac{g_A^4L(q_1)(c_2+c_3)(q_1z+q_3)}{128\pi^2F_\pi^6q_3\left(z^2-1\right)}\\
& +\frac{g_A^4L(q_3)(c_2+c_3)(q_1+q_3z)}{128\pi^2F_\pi^6q_1\left(z^2-1\right)},\\
\mathcal{R}_{6}^{(5),g_A^4} \;=\; & \mathcal{R}_{18}^{(5),g_A^4} \;=\; \mathcal{R}_{19}^{(5),g_A^4} \;=\; \mathcal{R}_{20}^{(5),g_A^4} \;=\; \mathcal{R}_{21}^{(5),g_A^4} \;=\; \mathcal{R}_{22}^{(5),g_A^4} \;=\; 0.
\end{aligned}
\tag{A.2}
$$

In the above expressions, q_1 and q_3 are always to be understood as the magnitudes of the corresponding three-momenta (except in the arguments of the function I), $q_1 \equiv |\vec{q}_1|$, $q_3 \equiv |\vec{q}_3|$. Further, the function $I(d:p_1,p_2,p_3;p_4)$ refers to the scalar loop integral

$$
I(d:p_1,p_2,p_3;p_4) \;=\; \frac{1}{i}\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l+p_1)^2-M_\pi^2+i\epsilon}\frac{1}{(l+p_2)^2-M_\pi^2+i\epsilon}\frac{1}{(l+p_3)^2-M_\pi^2+i\epsilon}\frac{1}{v\cdot(l+p_4)+i\epsilon}\,. \tag{A.3}
$$

This expression involves four-momenta p_i. For the case $p_i^0=0$ which we are interested in, it can be expressed in terms of the three-point function in Euclidean space $J\left(d:\vec{p}_1,\vec{p}_2,\vec{p}_3\right)$

$$
J\left(d:\vec{p}_1,\vec{p}_2,\vec{p}_3\right)=\int\frac{d^dl}{(2\pi)^d}\frac{1}{(\vec{l}+\vec{p}_1)^2+M_\pi^2}\frac{1}{(\vec{l}+\vec{p}_2)^2+M_\pi^2}\frac{1}{(\vec{l}+\vec{p}_3)^2+M_\pi^2}. \tag{A.4}
$$

In particular, the function $I\left(4:0,-q_1,q_3;0\right)$ which enters the expressions for R_i can be written as

$$
I\left(4:0,-q_1,q_3;0\right)=\frac{1}{2}J\left(3:\vec{0},-\vec{q}_1,\vec{q}_3\right). \tag{A.5}
$$

The N^{4}LO contributions to the structure functions proportional to g_A^2 vanish for $\mathcal{R}_{1,3,4,6,8,9,10,15,\ldots,22}$. The nonvan-

ishing contributions have the form

$$
\begin{aligned}
\mathcal{R}_2^{(5),g_A^2} &= -\frac{g_A^2}{128F_\pi^6\left(z^2-1\right)^2}I(4:0,-q_1,q_3;0)\left(32c_1\left(z^2-1\right)\left(4\left(z^2-1\right)M_\pi^2-q_1^2-q_3^2+2\left(q_1^2+q_3^2\right)z^2\right.\right.\\
&\left.+2q_1q_3z\right)M_\pi^2-8c_3\left(2M_\pi^2+q_1^2+q_3^2+2q_1q_3z\right)\left(z^2-1\right)\left(-4M_\pi^2-q_1^2-q_3^2+2\left(2M_\pi^2+q_1^2+q_3^2\right)z^2\right.\\
&\left.+2q_1q_3z\right)-c_2\left(-4M_\pi^2-q_1^2-q_3^2+4\left(M_\pi^2+q_1^2+q_3^2\right)z^2+6q_1q_3z\right)\\
&\left.\times\left(4\left(z^2-1\right)M_\pi^2-q_3^2-q_1(q_1+2q_3z)\right)\right)\\
&-\frac{g_A^2}{3072F_\pi^6\pi^2q_3\left(z^2-1\right)^2}L(q_1)\left(-96c_1\left(z^2-1\right)\left(q_1z+q_3\left(2z^2-1\right)\right)M_\pi^2+c_2\left(-3\left(2z^3+z\right)q_1^3\right.\right.\\
&\left.+q_3\left(8\left(z^2-5\right)z^2+5\right)q_1^2-3q_3^2z\left(4z^2+5\right)q_1+3q_3^3\left(1-4z^2\right)+4M_\pi^2\left(z^2-1\right)\left(3q_1z+q_3\left(8z^2-5\right)\right)\right)\\
&+8c_3\left(z^2-1\right)\left(3zq_1^3+q_3\left(16z^2-7\right)q_1^2+3q_3^2z\left(2z^2+1\right)q_1+3q_3^3\left(2z^2-1\right)\right.\\
&\left.\left.+M_\pi^2\left(6q_1z+q_3\left(28z^2-22\right)\right)\right)\right)\\
&-\frac{g_A^2}{3072F_\pi^6\pi^2q_1q_3\left(z^2-1\right)^2}L(q_2)\left(96c_1\left(z^2-1\right)\left(zq_1^2-2q_3\left(z^2-2\right)q_1+q_3^2z\right)M_\pi^2\right.\\
&-24c_3\left(2M_\pi^2+q_1^2+q_3^2+2q_1q_3z\right)\left(z^2-1\right)\left(zq_1^2-2q_3\left(z^2-2\right)q_1+q_3^2z\right)\\
&+c_2\left(\left(q_1^2+2q_3zq_1+q_3^2\right)\left(4q_1q_3z^4+6\left(q_1^2+q_3^2\right)z^3-2q_1q_3z^2+3\left(q_1^2+q_3^2\right)z+16q_1q_3\right)\right.\\
&\left.\left.-4M_\pi^2\left(z^2-1\right)\left(-10q_1q_3z^2+3\left(q_1^2+q_3^2\right)z+16q_1q_3\right)\right)\right)\\
&-\frac{g_A^2}{3072F_\pi^6\pi^2q_1\left(z^2-1\right)^2}L(q_3)\left(-96c_1\left(z^2-1\right)\left(q_3z+q_1\left(2z^2-1\right)\right)M_\pi^2+c_2\left(\left(3-12z^2\right)q_1^3\right.\right.\\
&-3q_3z\left(4z^2+5\right)q_1^2+q_3^2\left(8\left(z^2-5\right)z^2+5\right)q_1-3q_3^3z\left(2z^2+1\right)+4M_\pi^2\left(z^2-1\right)\\
&\left.\times\left(3q_3z+q_1\left(8z^2-5\right)\right)\right)+8c_3\left(z^2-1\right)\left(\left(6z^2-3\right)q_1^3+3q_3\left(2z^3+z\right)q_1^2+q_3^2\left(16z^2-7\right)q_1\right.\\
&\left.\left.+3q_3^3z+M_\pi^2\left(6q_3z+q_1\left(28z^2-22\right)\right)\right)\right)+\frac{c_2\left(q_1^2+2q_3zq_1+q_3^2\right)g_A^2}{1024F_\pi^6\pi^2\left(z^2-1\right)},\\
\mathcal{R}_5^{(5),g_A^2} &= \frac{g_A^2}{8F_\pi^6\left(z^2-1\right)}I(4:0,-q_1,q_3;0)c_4q_1z(q_1z+q_3)\left(q_1^2+2q_1q_3z+q_3^2\right)\\
&+\frac{g_A^2c_4L(q_2)(q_1z+q_3)\left(q_1^2+2q_1q_3z+q_3^2\right)}{64\pi^2F_\pi^6q_3\left(z^2-1\right)}-\frac{g_A^2c_4q_1zL(q_1)\left(q_1^2+2q_1q_3z+q_3^2\right)}{64\pi^2F_\pi^6q_3\left(z^2-1\right)}\\
&-\frac{g_A^2c_4L(q_3)\left(q_1^2z^2+q_1q_3z\left(z^2+1\right)+q_3^2\right)}{64\pi^2F_\pi^6\left(z^2-1\right)},\\
\mathcal{R}_7^{(5),g_A^2} &= -\frac{g_A^2}{64F_\pi^6q_1q_3\left(z^2-1\right)^2}I(4:0,-q_1,q_3;0)c_4z\left(q_1^2+2q_1q_3z+q_3^2\right)\left(4M_\pi^2\left(z^2-1\right)+q_1^2\left(2z^2-1\right)\right.\\
&\left.+2q_1q_3z+q_3^2\left(2z^2-1\right)\right)\\
&-\frac{g_A^2}{1536\pi^2F_\pi^6q_1^2q_3^2\left(z^2-1\right)^2}c_4L(q_2)\left(q_1^2+2q_1q_3z+q_3^2\right)\left(16M_\pi^2\left(z^2-1\right)+q_1^2\left(7z^2-4\right)\right.\\
&\left.+2q_1q_3z\left(z^2+2\right)+q_3^2\left(7z^2-4\right)\right)\\
&+\frac{g_A^2}{1536\pi^2F_\pi^6q_1q_3^2\left(z^2-1\right)^2}c_4L(q_1)\left(16M_\pi^2\left(z^2-1\right)(q_1+q_3z)+q_1^3\left(7z^2-4\right)+q_1^2q_3z\left(10z^2-1\right)\right.\\
&\left.+3q_1q_3^2z^2\left(2z^2+1\right)+3q_3^3z\left(2z^2-1\right)\right)\\
&+\frac{g_A^2}{1536\pi^2F_\pi^6q_1^2q_3\left(z^2-1\right)^2}c_4L(q_3)\left(16M_\pi^2\left(z^2-1\right)(q_1z+q_3)+3q_1^3z\left(2z^2-1\right)+3q_1^2q_3z^2\left(2z^2+1\right)\right.\\
&\left.+q_1q_3^2z\left(10z^2-1\right)+q_3^3\left(7z^2-4\right)\right)+\frac{g_A^2c_4\left(z\left(q_1^2+q_3^2\right)+2q_1q_3\right)}{1536\pi^2F_\pi^6q_1q_3\left(z^2-1\right)},\\
\mathcal{R}_{11}^{(5),g_A^2} &= \frac{g_A^2}{16F_\pi^6q_1\left(z^2-1\right)^2}I(4:0,-q_1,q_3;0)c_4z(q_1z+q_3)\left(3\left(q_1^2+2q_1q_3z+q_3^2\right)-4M_\pi^2\left(z^2-1\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{g_A^2 c_4\left(z^2+2\right) L(q_2)(q_1 z+q_3)\left(q_1^2+2 q_1 q_3 z+q_3^2\right)}{128 \pi^2 F_\pi^6 q_1^2 q_3\left(z^2-1\right)^2}-\frac{g_A^2 c_4 z L(q_1)\left(q_1^2\left(z^2+2\right)+6 q_1 q_3 z+3 q_3^2\right)}{128 \pi^2 F_\pi^6 q_1 q_3\left(z^2-1\right)^2} \\
&-\frac{g_A^2 c_4 L(q_3)\left(3 q_1^2 z^2+3 q_1 q_3 z\left(z^2+1\right)+q_3^2\left(z^2+2\right)\right)}{128 \pi^2 F_\pi^6 q_1^2\left(z^2-1\right)^2}-\frac{g_A^2 c_4(q_1+q_3 z)}{128 \pi^2 F_\pi^6 q_1\left(z^2-1\right)},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_{12}^{(5), g_A^2} = &-\frac{g_A^2}{16 F_\pi^6 q_3\left(z^2-1\right)^2} I(4: 0,-q_1, q_3 ; 0) c_4 z\left(\left(q_1^2+2 q_1 q_3 z+q_3^2\right)\left(2 q_1 z^2+q_1+3 q_3 z\right)\right. \\
&\left.-4 M_\pi^2\left(z^2-1\right)(q_1+q_3 z)\right)-\frac{g_A^2 c_4 z L(q_2)\left(q_1^2+2 q_1 q_3 z+q_3^2\right)\left(3 q_1 z+q_3\left(z^2+2\right)\right)}{128 \pi^2 F_\pi^6 q_1 q_3^2\left(z^2-1\right)^2} \\
&+\frac{g_A^2 c_4 z L(q_1)\left(3 z\left(q_1^2+q_3^2\right)+5 q_1 q_3 z^2+q_1 q_3\right)}{128 \pi^2 F_\pi^6 q_3^2\left(z^2-1\right)^2} \\
&+\frac{g_A^2 c_4 z L(q_3)\left(q_1^2\left(2 z^2+1\right)+2 q_1 q_3 z\left(z^2+2\right)+q_3^2\left(z^2+2\right)\right)}{128 \pi^2 F_\pi^6 q_1 q_3\left(z^2-1\right)^2}+\frac{g_A^2 c_4(q_1 z+q_3)}{128 \pi^2 F_\pi^6 q_3\left(z^2-1\right)},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_{13}^{(5), g_A^2} = &-\frac{g_A^2}{16 F_\pi^6 q_3\left(z^2-1\right)^2} I(4: 0,-q_1, q_3 ; 0) c_4(q_1 z+q_3)\left(\left(2 z^2+1\right)\left(q_1^2+2 q_1 q_3 z+q_3^2\right)-4 M_\pi^2\left(z^2-1\right)\right) \\
&-\frac{3 g_A^2 c_4 z L(q_2)(q_1 z+q_3)\left(q_1^2+2 q_1 q_3 z+q_3^2\right)}{128 \pi^2 F_\pi^6 q_1 q_3^2\left(z^2-1\right)^2}+\frac{g_A^2 c_4 L(q_1)\left(3 q_1^2 z^2+2 q_1 q_3\left(2 z^3+z\right)+q_3^2\left(2 z^2+1\right)\right)}{128 \pi^2 F_\pi^6 q_3^2\left(z^2-1\right)^2} \\
&+\frac{g_A^2 c_4 L(q_3)\left(q_1^2\left(2 z^3+z\right)+q_1 q_3\left(2 z^4+3 z^2+1\right)+3 q_3^2 z\right)}{128 \pi^2 F_\pi^6 q_1 q_3\left(z^2-1\right)^2}+\frac{g_A^2 c_4(q_1 z+q_3)}{128 \pi^2 F_\pi^6 q_3\left(z^2-1\right)},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_{14}^{(5), g_A^2} = &\ \frac{g_A^2}{16 F_\pi^6 q_3^2\left(z^2-1\right)^2} I(4: 0,-q_1, q_3 ; 0) c_4 q_1\left(\left(q_1^2+2 q_1 q_3 z+q_3^2\right)\left(2 q_1 z^2+q_1+3 q_3 z\right)\right. \\
&\left.-4 M_\pi^2\left(z^2-1\right)(q_1+q_3 z)\right)+\frac{g_A^2 c_4 L(q_2)\left(q_1^2+2 q_1 q_3 z+q_3^2\right)\left(3 q_1 z+q_3\left(z^2+2\right)\right)}{128 \pi^2 F_\pi^6 q_3^3\left(z^2-1\right)^2} \\
&-\frac{g_A^2 c_4 L(q_3)\left(q_1^2\left(2 z^2+1\right)+2 q_1 q_3 z\left(z^2+2\right)+q_3^2\left(z^2+2\right)\right)}{128 \pi^2 F_\pi^6 q_3^2\left(z^2-1\right)^2} \\
&-\frac{g_A^2 c_4 q_1 L(q_1)\left(3 z\left(q_1^2+q_3^2\right)+5 q_1 q_3 z^2+q_1 q_3\right)}{128 \pi^2 F_\pi^6 q_3^3\left(z^2-1\right)^2}-\frac{g_A^2 c_4 q_1(q_1+q_3 z)}{128 \pi^2 F_\pi^6 q_3^2\left(z^2-1\right)}.
\end{aligned}
\tag{A.6}
$$

Finally, the N^{4}LO contributions to the structure functions proportional to g_A^0 vanish for $\mathcal{R}_{1,3,\ldots,6,8,\ldots,22}$. The nonvanishing contributions have the form

$$
\begin{aligned}
\mathcal{R}_2^{(5), g_A^0} = &-\frac{1}{256 F_\pi^6\left(z^2-1\right)^2} I(4: 0,-q_1, q_3 ; 0)\left(4 M_\pi^2\left(z^2-1\right)-q_1(q_1+2 q_3 z)-q_3^2\right)\left(32 c_1 M_\pi^2\left(z^2-1\right)\right. \\
&\left.+3 c_2\left(-4 M_\pi^2\left(z^2-1\right)+q_1^2+2 q_1 q_3 z+q_3^2\right)-8 c_3\left(z^2-1\right)\left(2 M_\pi^2+q_1^2+2 q_1 q_3 z+q_3^2\right)\right) \\
&-\frac{1}{2048 \pi^2 F_\pi^6 q_1 q_3\left(z^2-1\right)^2} z L(q_2)\left(q_1^2+2 q_1 q_3 z+q_3^2\right)\left(-32 c_1 M_\pi^2\left(z^2-1\right)+c_2\left(20 M_\pi^2\left(z^2-1\right)\right.\right. \\
&\left.\left.+\left(2 z^2-5\right)\left(q_1^2+2 q_1 q_3 z+q_3^2\right)\right)+8 c_3\left(z^2-1\right)\left(2 M_\pi^2+q_1^2+2 q_1 q_3 z+q_3^2\right)\right) \\
&-\frac{1}{6144 \pi^2 F_\pi^6 q_3\left(z^2-1\right)^2} L(q_1)\left(96 c_1 M_\pi^2\left(z^2-1\right)(q_1 z+q_3)+3 c_2(q_1 z+q_3)\left(-20 M_\pi^2\left(z^2-1\right)\right.\right. \\
&\left.+q_1^2\left(5-2 z^2\right)+6 q_1 q_3 z+3 q_3^2\right)-8 c_3\left(z^2-1\right)\left(2 M_\pi^2\left(3 q_1 z+q_3\left(7-4 z^2\right)\right)+3 q_1^3 z\right. \\
&\left.\left.+q_1^2 q_3\left(4 z^2+5\right)+9 q_1 q_3^2 z+3 q_3^3\right)\right) \\
&-\frac{1}{6144 \pi^2 F_\pi^6 q_1\left(z^2-1\right)^2} L(q_3)\left(96 c_1 M_\pi^2\left(z^2-1\right)(q_1+q_3 z)+3 c_2(q_1+q_3 z)\left(-20 M_\pi^2\left(z^2-1\right)\right.\right. \\
&\left.+3 q_1^2+6 q_1 q_3 z+q_3^2\left(5-2 z^2\right)\right)-8 c_3\left(z^2-1\right)\left(2 M_\pi^2\left(q_1\left(7-4 z^2\right)+3 q_3 z\right)+3 q_1^3+9 q_1^2 q_3 z\right. \\
&\left.\left.+q_1 q_3^2\left(4 z^2+5\right)+3 q_3^3 z\right)\right)-\frac{c_2\left(q_1^2+2 q_1 q_3 z+q_3^2\right)}{2048 \pi^2 F_\pi^6\left(z^2-1\right)},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_7^{(5),g_A^0} \;=\;& \frac{1}{128F_\pi^6 q_1 q_3 \left(z^2-1\right)^2} I(4:0,-q_1,q_3;0) c_4 z \left(q_1^2+2q_1q_3z+q_3^2\right)\left(-4M_\pi^2\left(z^2-1\right)+q_1^2+2q_1q_3z+q_3^2\right) \\
&+\frac{c_4 L(q_2)\left(q_1^2+2q_1q_3z+q_3^2\right)\left(\left(z^2+2\right)\left(q_1^2+2q_1q_3z+q_3^2\right)-8M_\pi^2\left(z^2-1\right)\right)}{3072\pi^2F_\pi^6 q_1^2 q_3^2\left(z^2-1\right)^2} \\
&-\frac{c_4 L(q_1)\left(-8M_\pi^2\left(z^2-1\right)\left(q_1+q_3z\right)+q_1^3\left(z^2+2\right)+q_1^2q_3z\left(4z^2+5\right)+9q_1q_3^2z^2+3q_3^3z\right)}{3072\pi^2F_\pi^6 q_1 q_3^2\left(z^2-1\right)^2} \\
&-\frac{c_4 L(q_3)\left(-8M_\pi^2\left(z^2-1\right)\left(q_1z+q_3\right)+3q_1^3z+9q_1^2q_3z^2+q_1q_3^2z\left(4z^2+5\right)+q_3^3\left(z^2+2\right)\right)}{3072\pi^2F_\pi^6 q_1^2 q_3\left(z^2-1\right)^2} \\
&-\frac{c_4\left(z\left(q_1^2+q_3^2\right)+2q_1q_3\right)}{3072\pi^2F_\pi^6 q_1 q_3\left(z^2-1\right)}. \qquad \text{(A.7)}
\end{aligned}
$$

A Mathematica notebook which contains the above expressions for the structure functions in momentum space is available from the authors upon request.

---

[1] E. Stephan, S. Kistryn, R. Sworst, A. Biegun, K. Bodek, I. Ciepal, A. Deltuva and E. Epelbaum *et al.*, Phys. Rev. C **82**, 014003 (2010).
[2] I. Ciepal, S. .Kistryn, E. Stephan, A. Biegun, K. Bodek, A. Deltuva, E. Epelbaum and M. Eslami-Kalantari *et al.*, Phys. Rev. C **85**, 017001 (2012).
[3] M. Viviani, L. Girlanda, A. Kievsky, L. E. Marcucci and S. Rosati, arXiv:1004.1306 [nucl-th].
[4] P. Navratil, V. G. Gueorguiev, J. P. Vary, W. E. Ormand and A. Nogga, Phys. Rev. Lett. **99**, 042501 (2007) [nucl-th/0701038].
[5] D. Gazit, S. Quaglioni and P. Navratil, Phys. Rev. Lett. **103**, 102502 (2009) [arXiv:0812.4444 [nucl-th]].
[6] R. Roth, J. Langhammer, A. Calci, S. Binder and P. Navratil, Phys. Rev. Lett. **107**, 072501 (2011) [arXiv:1105.3173 [nucl-th]].
[7] K. Hebeler, J. M. Lattimer, C. J. Pethick and A. Schwenk, Phys. Rev. Lett. **105**, 161102 (2010) [arXiv:1007.1746 [nucl-th]].
[8] G. Hagen, M. Hjorth-Jensen, G. R. Jansen, R. Machleidt and T. Papenbrock, arXiv:1202.2839 [nucl-th].
[9] J. D. Holt, J. Menendez and A. Schwenk, Phys. Rev. Lett. **110**, 022502 (2013) [arXiv:1207.1509 [nucl-th]].
[10] I. Tews, T. Krueger, K. Hebeler and A. Schwenk, arXiv:1206.0025 [nucl-th].
[11] N. Kalantar-Nayestanaki, E. Epelbaum, J. G. Messchendorp and A. Nogga, Rept. Prog. Phys. **75**, 016301 (2012) [arXiv:1108.1227 [nucl-th]].
[12] H.-W. Hammer, A. Nogga and A. Schwenk, arXiv:1210.4273 [nucl-th].
[13] E. Epelbaum, H. -W. Hammer and U.-G. Meißner, Rev. Mod. Phys. **81**, 1773 (2009) [arXiv:0811.1338 [nucl-th]].
[14] R. Machleidt and D. R. Entem, Phys. Rept. **503**, 1 (2011) [arXiv:1105.2919 [nucl-th]].
[15] E. Epelbaum and U.-G. Meißner, Ann. Rev. Nucl. Part. Sci. **62**, 159 (2012) [arXiv:1201.2136 [nucl-th]].
[16] D. R. Entem and R. Machleidt, Phys. Rev. C **68**, 041001 (2003) [nucl-th/0304018].
[17] E. Epelbaum, W. Glöckle and U.-G. Meißner, Nucl. Phys. A **747**, 362 (2005) [nucl-th/0405048].
[18] U. van Kolck, Phys. Rev. C **49**, 2932 (1994).
[19] E. Epelbaum, A. Nogga, W. Glöckle, H. Kamada, U.-G. Meißner and H. Witała, Phys. Rev. C **66**, 064001 (2002) [nucl-th/0208023].
[20] E. Epelbaum, H. Krebs, D. Lee and U.-G. Meißner, Eur. Phys. J. A **41**, 125 (2009) [arXiv:0903.1666 [nucl-th]].
[21] V. Bernard, N. Kaiser and U.-G. Meißner, Nucl. Phys. A **615**, 483 (1997) [hep-ph/9611253].
[22] N. Fettes, U.-G. Meißner and S. Steininger, Nucl. Phys. A **640**, 199 (1998) [hep-ph/9803266].
[23] N. Fettes and U.-G. Meißner, Nucl. Phys. A **676**, 311 (2000) [hep-ph/0002162].
[24] P. Buettiker and U.-G. Meißner, Nucl. Phys. A **668**, 97 (2000) [hep-ph/9908247].
[25] J. M. Alarcon, J. M. Camalich and J. A. Oller, arXiv:1210.4450 [hep-ph].
[26] Y.-H. Chen, D.-L. Yao and H. Q. Zheng, arXiv:1212.1893 [hep-ph].
[27] M. C. M. Rentmeester, R. G. E. Timmermans and J. J. de Swart, Phys. Rev. C **67**, 044001 (2003) [nucl-th/0302080].
[28] V. Bernard, E. Epelbaum, H. Krebs and U.-G. Meißner, Phys. Rev. C **77**, 064004 (2008) [arXiv:0712.1967 [nucl-th]].
[29] S. Ishikawa and M. R. Robilotta, Phys. Rev. C **76**, 014006 (2007) [arXiv:0704.0711 [nucl-th]].
[30] V. Bernard, E. Epelbaum, H. Krebs and U.-G. Meißner, Phys. Rev. C **84**, 054001 (2011) [arXiv:1108.3816 [nucl-th]].
[31] M. R. Robilotta, Phys. Rev. C **74**, 044002 (2006) [Erratum-ibid. C **74**, 059902 (2006)] [nucl-th/0610046].
[32] J. Golak, D. Rozpedzik, R. Skibinski, K. Topolnicki, H. Witała, W. Glöckle, A. Nogga and E. Epelbaum *et al.*, Eur. Phys. J. A **43**, 241 (2010) [arXiv:0911.4173 [nucl-th]].
[33] R. Skibinski, J. Golak, K. Topolnicki, H. Witała, E. Epelbaum, W. Glöckle, H. Krebs and A. Nogga *et al.*, Phys. Rev. C **84**, 054005 (2011) [arXiv:1107.5163 [nucl-th]].
[34] C. Ordonez, L. Ray and U. van Kolck, Phys. Rev. Lett. **72**, 1982 (1994).

[35] N. Kaiser, S. Gerstendorfer and W. Weise, Nucl. Phys. A **637**, 395 (1998) [nucl-th/9802071].
[36] H. Krebs, E. Epelbaum and U.-G. Meißner, Eur. Phys. J. A **32**, 127 (2007) [nucl-th/0703087].
[37] N. Kaiser, R. Brockmann and W. Weise, Nucl. Phys. A **625**, 758 (1997) [nucl-th/9706045].
[38] R. Machleidt and D. R. Entem, J. Phys. G G **37**, 064041 (2010) [arXiv:1001.0966 [nucl-th]].
[39] S. C. Pieper, V. R. Pandharipande, R. B. Wiringa and J. Carlson, Phys. Rev. C **64**, 014001 (2001) [nucl-th/0102004].
[40] H. Krebs, A. Gasparyan and E. Epelbaum, Phys. Rev. C **85**, 054006 (2012) [arXiv:1203.0067 [nucl-th]].
[41] N. Fettes, U.-G. Meißner, M. Mojzis and S. Steininger, Annals Phys. **283**, 273 (2000) [Erratum-ibid. **288**, 249 (2001)] [hep-ph/0001308].
[42] E. Epelbaum, U.-G. Meißner and J. E. Palomar, Phys. Rev. C **71**, 024001 (2005) [nucl-th/0407037].
[43] W. Glöckle, *The Quantum Mechanical Few-Body Problem*, Springer-Verlag, 1983.
[44] V. Baru, E. Epelbaum, C. Hanhart, M. Hoferichter, A. E. Kudryavtsev and D. R. Phillips, Eur. Phys. J. A **48**, 69 (2012) [arXiv:1202.0208 [nucl-th]].
[45] L. Girlanda, A. Kievsky and M. Viviani, Phys. Rev. C **84**, 014001 (2011) [arXiv:1102.4799 [nucl-th]].
[46] C. Ordonez, L. Ray and U. van Kolck, Phys. Rev. C **53**, 2086 (1996) [hep-ph/9511380].
[47] E. Epelbaum, H. Krebs and U.-G. Meißner, Nucl. Phys. A **806**, 65 (2008) [arXiv:0712.1969 [nucl-th]].
[48] E. Epelbaum, H. Krebs and U.-G. Meißner, Phys. Rev. C **77**, 034006 (2008) [arXiv:0801.1299 [nucl-th]].